A Compact Enharmonically Viable Subset of Harmonic Space

The Stern-Brocot Tree and Some Thoughts About Lattices and Spirals

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Abstract: While temperament may present some apparent advantages, it necessarily sacrifices the distinct "JI fusion sonority" of intervals that are *precisely* tuneable by ear. These manifest a vast gradation of consonance and dissonance, as yet largely unexplored and with fascinating potential for new compositions. This article proposes a novel approach, based on a mathematical structure called the Stern-Brocot Tree, allowing a finite set of pitches to be generated in *just intonation* within which music may modulate freely without relying on temperament.

Keywords: just intonation, microtonality, tuneability, harmonic space, logarithmic spiral, lattice, Stern-Brocot tree

Introduction

For over 500 years, Western music theory has postulated the necessity of temperament. This principle accepts some deliberate, irrational mistunings from just intonation (JI).[1] Music tuned in a tempered system may move freely between different tonal centres, each tuned to a consistent, fixed frequency, all the while giving an *impression* of changing harmony. Notes may suggest a multiplicity of relationships without introducing undesirable dissonances or an infinite proliferation of pitches.

Though it may present some apparent advantages, temperament sacrifices the distinct "JI fusion sonority" characteristic of intervals that are *precisely* tuneable by ear. This direct, psychoacoustic *sensation* of harmony ranges from the resonant, beatless smoothness of the simplest relationships to a distinct "periodic" snarling typical of higher prime-number intervals. JI extended in this way manifests a vast gradation of consonance and dissonance, as yet largely unexplored and with fascinating potential for new compositions. This article proposes a novel approach to generating a finite, evenly distributed set of pitches in just intonation within which music may modulate freely without relying on temperament.

Since no ratios other than powers of 2 (unisons, octaves) multiply to produce another power of 2, no finite JI pitch set can replicate the same non-power-of-2 interval at every note;[2] however, acoustic instruments and human hearing do allow a certain tolerance. In our experience, acoustic instruments, especially strings, are able to tune notes to within a schisma (2c) in critical harmonic contexts meant to be perceived as extended JI. Finer resolutions, though clearly possible with computers and electronics, confront human limitations of breath and bowing, as well as the nonlinearities of physical materials, all of which introduce small, random fluctuations.

Thus, some finite pitch sets may function as tolerably "complete" approximations of harmonic auditory cognition.[3] In such cases, when combining intervals it is desirable to maintain "consistency": ratios represented by best-case mistunings ought to combine to match best-case representations. Also, it is generally desirable to minimise the retuning of held notes in common-tone chord changes. These requirements cause adherents of adaptive JI[4] and quasi-JI equal tempered divisions to investigate extremely fine-grained pitch sets in the thousands of pitches.

1 JI or rational intonation is the practice of establishing the tuning of pitches in relation to each other, based on rational number ratios of frequencies, primarily using smaller numbers to maximise the sensation of "in-tuneness" or psychoacoustic harmonicity.

2 If the same interval taken from each note of a finite pitch set with *n* elements always produces another note in the set, then a finite number of these intervals must eventually produce a cycle of at most *n* notes returning to an exact unison with the starting pitch, transposed by some number of octaves. Mathematically, this is only possible if the interval is some power of 2 to begin with.

3 This term, coined by Marcus Pal, refers to the mechanism(s) by which we perceive harmonic relations between frequencies.

4 Adaptive JI, first proposed by Nicola Vicentino, is a form of temperament in which chords are tuned in just intonation, but melodic intervals are tempered.

This article proposes a different approach, one that has the potential to be tuneable by ear. It opens the door to the viability of enharmonic exchange (i.e. *tonality flux*[5] at the limits of tonal differentiation), allowing continuous traversal of the tonal network without requiring one to backtrack. A pitch set generated with this in mind may serve as a relatively "complete" depiction of the wide spectrum of harmonic experience as a fundamentally psychoacoustic and physiological experience.

Prioritising JI sonority requires a JI notation readable in real time, which necessarily limits the spectrum of available pitches simply due to the constraints of human understanding. Experience has shown that combinations of up to three Helmholtz-Ellis JI Pitch Notation (HEJI) symbols in an accidental string represent a practical complexity boundary, allowing substantial compositional exploration and real-time music making by human performers on acoustic instruments. Since its conception in 2000, the HEJI notation has become a standard for notating music in just intonation, and in 2020 the notation has been revised, extending the defined set of accidentals to the 47-limit (see below).

Some further desirable conditions for an extended but finite JI pitch set may include a relative degree of evenness, a multiplicity of possible tonal centres, and excellent representations of the aforementioned JI fusion sonority. Formally mapping such harmonic relationships begins with extending Leonhard Euler's "Tonnetz" through James Tenney's "Harmonic Space". This model integrates "pitch-height distance" with a quantitative measure of so-called harmonic distance within a single multidimensional lattice, extensible to any prime limit.

The approach we have developed makes use of an algorithm that generates and orders fractions in a manner analogous to harmonic distance. A set of lowest-terms rational numbers was formed using the Stern-Brocot Tree up to order 9, then these ratios were transposed to each of the ratios of the Pythagorean pentatonic, merged, and normalised; finally, duplicates were eliminated, obtaining a set of 933 pitches. This set, named 933-Tone-Tree Harmonic Space, brings together the various advantages of adaptive just intonation, fine-grain equal temperaments, and the unmediated sonic clarity and playability of rational intonation.

[5] A term adopted by Harry Partch to describe microtonal voice leading between two or more overtonal/undertonal harmonic structures in the same register. See Partch, *Genesis of a Music: An Account of a Creative Work, Its Roots, and Its Fulfillments*, second edition (Boston: Da Capo Press, 1979), 188–190.

Part 1: Methods of Managing and Understanding Harmony

The Euler Lattice and the syntonic comma

In 1739, Leonhard Euler conceived a simple lattice diagram to represent the structural harmonic space of music as he knew it – an inverted tree of 12 musical notes or pitch classes, generated from one fundamental reference, forming a network of interlocking major and minor triads.[6]

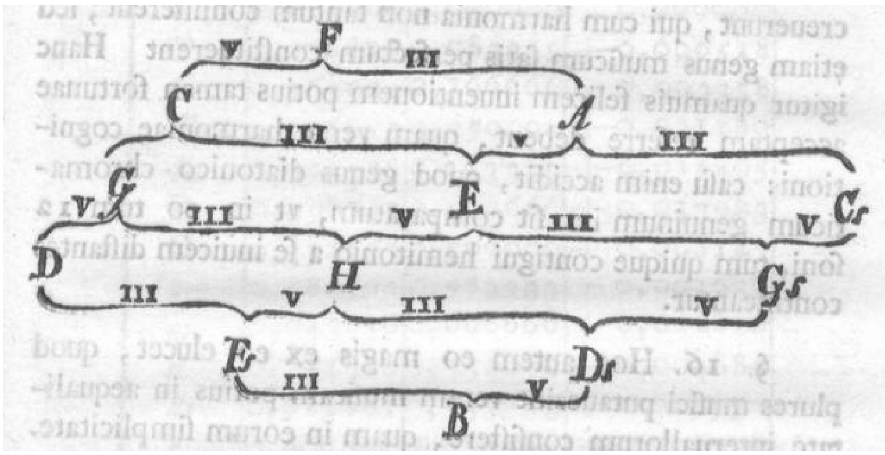


Figure 1: Note that the subscript ***s*** means "sharp", ***H*** refers to ♮B, and ***B*** refers to ♭B.

Beginning with ***F*** in the first row, each note spawns two children produced from two intervals, an ascending $\frac{3}{2}$ perfect fifth[7] and an ascending $\frac{5}{4}$ major third; both are written in the next row. These pitch-triples, linked by brackets, generate major triads in the frequency proportion 4 : 5 : 6. By construction, neighbouring notes in each row are a descending minor third apart.

Leftward diagonals indicate successive ascending perfect fifths $\left(\frac{3}{2}\right)$ and rightward diagonals indicate successive ascending major thirds $\left(\frac{5}{4}\right)$. The diagram's upper half generates the seven diatonic notes, tuned in a Lydian mode ***F–G–A–H–C–D–E–F***. Major triads rooted on ***F***, ***C***, and ***G***, respectively, form similar upward-pointing triangles. Downward-pointing triangles outline minor triads.

[6] Leonhard Euler, *Tentamen Novae Theoriae Musicae Ex Certissimis Harmoniae Principiis Dilucide Expositae* (ex typographia Academiae scientiarum, 1739), 147.

[7] As a convention, the authors distinguish ratios used to describe relative *intervals*, which are written in the form $\frac{b}{a}$ or $a : b$, and those representing absolute *pitches*, which are written as fractions b/a.

This subset of diatonic pitches contains two different whole tones, each with its own geometric relationship in Euler's lattice. ***F–G***, ***A–H***, and ***C–D*** are each composed of two steps downward-left, comprising two ascending fifths to produce the normalised[8] ratio $\frac{9}{8}$. ***G–A*** and ***D–E*** are composed of one step upward-right and one step horizontal-right, comprising a descending fifth and minor third. These combine to produce the normalised ratio $\frac{10}{9}$. The difference between the two whole tones, $\frac{9}{8}$ and $\frac{10}{9}$, is the *syntonic comma* with ratio $\frac{81}{80}$(ca. 21.5 cents).

The intonation of these diatonic notes follows the interval ratios of the Ptolemaic tense diatonic ($\frac{10}{9}$, $\frac{9}{8}$, $\frac{16}{15}$) in each of the descending tetrachords ***E–D–C–H*** and ***A–G–F–E***. When taken in the mode ***C–C*** ascending this sequence is often referred to as the Ptolemaic or 5-limit major scale in just intonation.

It is curious to note the unusual position of the note ***B***, representing ♭B. Chromatically raised notes ***Fs***, ***Cs***, ***Gs***, and ***Ds*** are each positioned in their logical place as extensions from the single generating reference pitch ***F*** at the top right. Where ***B*** is written, it is clear that the note ***As*** (i.e. ♯A) must fall; Euler writes ***B***, while implying that this 12th note might need to be intonated as ***As***. In his quantitative discussion of the pitch set, he indeed categorises the interval made by ***Fs*** and ***B*** as $\frac{5}{4}$.

One might surmise that he is simply choosing to respect the historically defined *musica vera* gamut of eight named notes along the series of fifths from ***B*** (*molle*, "soft") to ***H*** (*durum*, "hard"). On the other hand, Euler was a well-known advocate of including septimal intervals in harmonic progressions (a position later taken up by Giuseppe Tartini). The choice of ***As*** written as ***B*** implies a tuning that is a near-enharmonic proximity of $\frac{7}{4}$ above ***C***, varying from it by the small interval 224 : 225 or ca. 7.7 cents.

There are two other possible locations for ***B***, should this segment of the lattice be extended. These options would make it a parent either of ***F*** or of ***D***, and each of these "enharmonic" variations would necessitate a *new intonation*, as different possible positions in the lattice imply multiple and distinct geneses. The ***B*** one row above ***F*** would be one syntonic comma lower than the ***B*** one column to the left of ***D***, which itself is one diesis $\frac{128}{125}$ above the implied ***As*** originally written by Euler as ***B***. Each of these three intonations readily occurs in common musical practice, and their coexistence serves as a window into the enharmonic microtonality implicit in even the simplest JI rendering of the dodecaphonic pitch set.

The potential for infinite extension of the lattice in all directions is one of the characteristics of harmonic modulation in extended just intonation. After Euler, subsequent theorists, including Arthur von Oettingen and Hugo Riemann, returned to his lattice diagram and expanded it, valuing the clarity with which it reveals the multidimensional interconnectivity

8 A *normalised* or octave-reduced ratio is transposed by one or more octaves, if need be, such that it lies between 1/1 and 2/1; see discussion below.

of triadic, 5-limit harmony. The "Euler Lattice" eventually became known as the *Tonnetz* ("tone network").

Moritz Hauptmann and Hermann von Helmholtz advocated for notating the lattice's microtonal "difference[s] of intonation with certainty and without ambiguity".[9] Using the Pythagorean framework as a backbone, Helmholtz defined a system of sub- and superscripts to indicate lowering and raising tones by the number of syntonic commas needed to differentiate tones generated by means of major thirds rather than perfect fifths. Thus, he wrote the major third above F as "A_1" and the major third above "A_1" as "$C_2\sharp$", whereas the "A" and "C♯" in the chain of perfect fifths from F were written without any subscript. This system allowed for the precise notation of any pitch in 5-limit harmonic space.

While Helmholtz's sub- and superscript notation works effectively in the alphanumeric context of a book or an article, his system requires adaptation to produce a staff notation for music that seeks to make explicit this microtonal difference. Helmholtz argued against higher primes, while acknowledging that some such intervals were consonant, making the case that because of their inherently lesser consonance, their treatment would go against conventional practices of freely inverting and voicing chords in all registers. Nevertheless, by radically choosing to include ratios built upon 11° and 13°, early 20th-century composers Elsie Hamilton and Harry Partch extended the concept of just intonation beyond the 5-limit.[10] Inevitably, this extended just intonation progressively brings about more small adjustments like the syntonic comma; there is at least one distinctive comma associated with each new prime factor. Because of this, any staff notation would necessitate a system of symbols that is both unambiguous and expandable. One attempt at devising such a system is described in the next section.

Notating music in just intonation

The Extended Helmholtz-Ellis JI Pitch Notation (HEJI) was devised by Marc Sabat and Wolfgang von Schweinitz in an ongoing collaboration that began in Berlin, Germany, in 1999. At the time, both composers were determined to write instrumental music with respect to just intonation. They both imagined notating microtonal pitches in a way that would bring out unique psychoacoustic experiences of harmonic auditory cognition (how our minds perceive and process relations of tones). These include sensations of fusion, combination tones, virtual fundamentals, common partials, etc. Sabat and von Schweinitz surmised that an awareness of such sensory qualities and their interaction with the techniques of orchestration and instrumental realisation (playability), as well as an exact

9 Hermann von Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, trans. Alexander J. Ellis, third English edition (New York: Longmans, Green & Co., 1895), 276.

10 Shorthand notations combining a raised o or preceding u, i.e. 13° and u13, are to be read "13th overtone" or "over-13" and "13th undertone" or "under-13", respectively.

rather than approximate notation, would allow a greatly extended repertoire of tuned sounds – pitches, intervals, and aggregates – to emerge, thereby establishing a practice based on an empirically sensible continuum of *relative consonance and dissonance*.

Mindful of approaches taken by previous generations – Harry Partch's continuation of the ratio notation found in theoretical treatises since ancient times after various unsuccessful staff notation experiments; Ben Johnston's accidental system based on the 5-limit; James Tenney's combination of cents deviations with tempered pitch classes and approximate arrows; various tempered divisions of the tone into multiple equal parts used by early microtonalists and the spectral movement – Sabat and von Schweinitz chose to base their notation on the 19th-century work of Helmholtz and his translator, Alexander J. Ellis, who had drawn on contemporary work by Hauptmann and Oettingen (remarked on in the previous section). Namely, they sought to reawaken the Pythagorean system *implicit in the five-line staff notation* and to define higher primes as simple epimoric[11] alterations of nearby Pythagorean intervals by introducing corresponding symbols.

In the first version of their notation, published in 2001, this approach was followed strictly up to the 16th partial, since the lower primes require substantial and readily perceptible alterations from the Pythagorean. Above 16°, the alterations are either smaller (i.e. 17° and 19°) or the partials themselves gradually exceed the limits of intervallic saliency when presented as simultaneous intervals (see discussion below). Therefore, primes from 17° to 31° were originally notated with respect to nearby 13-limit sounds; 17° and 29° altered ratios $\frac{16}{15}$ and $\frac{9}{5}$, respectively, 31° altered $\frac{64}{33}$, while 19° and 23° were written with respect to the Pythagorean. Beyond 31°, combinations of lower primes could always provide near-enharmonic substitutions, which were used to notate higher primes, written by enclosing the accidentals in curly brackets. For example, the prime 47° was notated as an inversion of the notation for partial 49° (7°×7°) with curly brackets to differentiate it.

As the notation has come into common practice, various musicians working with it have expressed a wish that it more strictly respect its Pythagorean underpinning, even with respect to the less "tuneable" higher primes, and provide distinctive logos further up the series. In our 2020 revision, created in collaboration with von Schweinitz, Catherine Lamb, and M.O. Abbott, HEJI was updated and expanded through the 47-limit. Additionally, a new principle of extensibility was also introduced, where, on a case-by-case basis, symbols outside the defined gamut may be associated with ratios to simplify notation of certain accidental combinations. HEJI provides three such symbols to serve as an example of how this principle might be implemented. For example, the schisma is notated by means of a raised or lowered tilde, showing the near-enharmonic intersection of 8 ascending perfect fifths and one descending major third. The other two symbols simplify the combinations of 11° and 13°.

[11] An epimoric or superparticular ratio is one where the numerator exceeds the denominator by one, representing, for example, the interval between two neighbouring harmonic partials.

In its standard form, HEJI notation has the following basic characteristics:

– Every note is written with *at least one* accidental, even if it is simply a natural sign;
– The alteration for prime 5 (arrow-up, arrow-down, etc.) is the only symbol that forms a ligature with the Pythagorean accidentals (… 𝄫, ♭, ♮, ♯, 𝄪 …) and with itself (resulting in combinations like ♮ and ♭, etc.); the only exception is the double alteration for prime 7 (i.e. 49°), which forms a ligature with itself (e.g. + =) but not with the Pythagorean accidentals;
– All other prime alterations are notated as an ordered string of symbols in the standard position to the left of the notehead such that the symbol of the smallest prime is closest to the notehead and that of the largest is farthest;
– For alterations of a Pythagorean natural by primes beyond 5, the natural symbol may be left out;
– To facilitate reading multiple accidentals, the locally less significant signs (i.e. those that indicate the shifted fundamental and are thus replicated on all its partials) may be reduced in size if desired;
– Cent deviations may be indicated in conjunction with the accidentals; positive deviations may be placed just above the accidental, and negative deviations may be placed below, though this is not always practical nor necessary; cent deviations are generally indicated in relation to the nearest tempered pitch class as would be read on a tuning meter; unless otherwise indicated, the pitch with 0 deviation is generally the Kammerton A440 notated as Pythagorean A; if the nearest pitch class differs from the notated one, it may be added to the cents string as text.

A legend summarising the notation as well as an example comprising the harmonic series of A0 (27.5 Hz) and the subharmonic series of E7 (2640 Hz) are included for reference below. Portrait and landscape formats of these reference documents, as well as fonts and various other resources, are available under an open-source license from Plainsound Music Edition (plainsound.org). These may be used freely as reference material in scores. For idiosyncratic uses of the notation that go outside these standard parameters, it is suggested to include an appropriate explanatory note alongside these documents.

The Helmholtz-Ellis JI Pitch Notation (HEJI) | 2020 | LEGEND

revised by Marc Sabat and Thomas Nicholson | PLAINSOUND MUSIC EDITION | www.plainsound.org

in collaboration with Wolfgang von Schweinitz, Catherine Lamb, and M.O. Abbott, building upon the original HEJI notation devised by Marc Sabat and Wolfgang von Schweinitz in the early 2000s

PYTHAGOREAN JUST INTONATION | generated by multiplying / dividing an arbitrary reference frequency by PRIMES 2 and 3 only

… 𝄫 ♭ ♮ ♯ 𝄪 …

notate a series of ***perfect fifths*** *above / below a reference*
$^{3}/_{2} \approx \pm 702.0$ cents *(i.e. 2c wider than tempered)*
each new accidental represents 7 fifths, altering by one apotome
$^{2187}/_{2048} \approx \pm 113.7$ cents

Frequency ratios including higher prime numbers (5–47) may be notated by adding the following distinct accidental symbols. Custom indications for higher primes or various enharmonic substitutions may be invented as needed by simply defining further symbols representing the relevant ratio alterations.

PTOLEMAIC JUST INTONATION | PRIMES up to 5

includes the consonant ***just major third***
$^{5}/_{4} \approx \pm 386.3$ cents *(ca. 14c narrower than tempered)*

alteration by one syntonic comma
$^{81}/_{80} \approx \pm 21.5$ cents

alteration by two syntonic commas
$^{81}/_{80} \cdot {}^{81}/_{80} \approx \pm 43.0$ cents

~ = ♭   ~ = ♯

alteration by one schisma to notate an exact enharmonic substitution
$^{32805}/_{32768} \approx \pm 2.0$ cents

SEPTIMAL JI | PRIME 7

includes the consonant ***natural seventh***
$^{7}/_{4} \approx \pm 968.8$ cents *(ca. 31c narrower than tempered)*

alteration by one septimal comma (Giuseppe Tartini)
$^{64}/_{63} \approx \pm 27.3$ cents

alteration by two septimal commas
$^{64}/_{63} \cdot {}^{64}/_{63} \approx \pm 54.5$ cents

UNDECIMAL | PRIME 11

includes the ***undecimal semi-augmented fourth***
$^{11}/_{8} \approx \pm 551.3$ cents *(ca. 51c wider than tempered)*

alteration by one undecimal quartertone (Richard H. Stein)
$^{33}/_{32} \approx \pm 53.3$ cents

TRIDECIMAL | PRIME 13

includes the ***tridecimal neutral sixth***
$^{13}/_{8} \approx \pm 840.5$ cents *(ca. 59c narrower than a tempered major sixth)*

alteration by one tridecimal thirdtone (Gérard Grisey)
$^{27}/_{26} \approx \pm 65.3$ cents

≈♭ =   ≈♯ =

combination of $^{11}/_{13}$ *re-notated enharmonically*
alteration by the ratio $^{352}/_{351} \approx \pm 4.9$ cents

♮ =   ♮ =

*combination of 11 * 13 re-notated as a single symbol*
alteration by the ratio $^{144}/_{143} \approx \pm 12.1$ cents

PRIMES 17 THROUGH 47

alteration by one 17-limit schisma
$^{2187}/_{2176} \approx \pm 8.7$ cents

alteration by one 19-limit schisma
$^{513}/_{512} \approx \pm 3.4$ cents

↓ ↑ *alteration by one 23-limit comma (James Tenney / John Cage)*
$^{736}/_{729} \approx \pm 16.5$ cents

⇓ ⇑ *alteration by one 29-limit sixthtone*
$^{261}/_{256} \approx \pm 33.5$ cents

alteration by one 31-limit quartertone (Alinaghi Vaziri)
$^{32}/_{31} \approx \pm 55.0$ cents

alteration by one 37-limit quartertone (Ivan Wyschnegradsky)
$^{37}/_{36} \approx \pm 47.4$ cents

− + *alteration by one 41-limit comma (Ben Johnston)*
$^{82}/_{81} \approx \pm 21.2$ cents

▾ ▴ *alteration by one 43-limit comma*
$^{129}/_{128} \approx \pm 13.5$ cents

alteration by one 47-limit quartertone
$^{752}/_{729} \approx \pm 53.8$ cents

CENTS HEJI accidentals may be combined with an indication of their deviation in cents from equal temperament as read on a tuning meter; A♮ 440 Hz is usually defined to be ±0 cents. If this deviation exceeds ±50 cents, the nearest tempered pitch-class may be added: e.g. A (−65 cents from A♮) could include the annotation A♭+35 placed above or below its accidental.

TEMPERED NOTES | may be combined with cents deviations to notate free microtonal pitches

… 𝄫 ♭ ♮ ♯ 𝄪 …

indicate the respective equal tempered quartertone;
show which pitch is assigned a deviation of 0c

The Helmholtz-Ellis JI Pitch Notation (HEJI) | 2020

Harmonic / Subharmonic series 1–49 notated by modifications of Pythagorean notes

with dedicated microtonal accidental symbols for primes 5 through 47

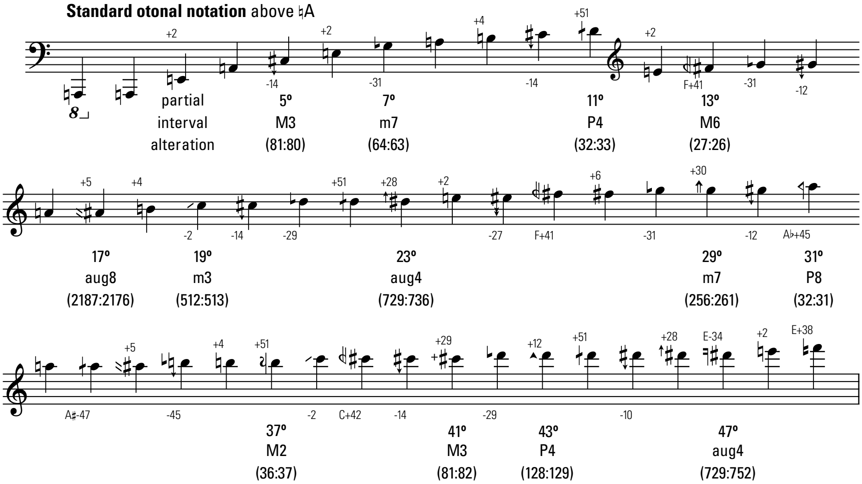
revised by Marc Sabat & Thomas Nicholson

in collaboration with Wolfgang von Schweinitz, Catherine Lamb and M.O. Abbott

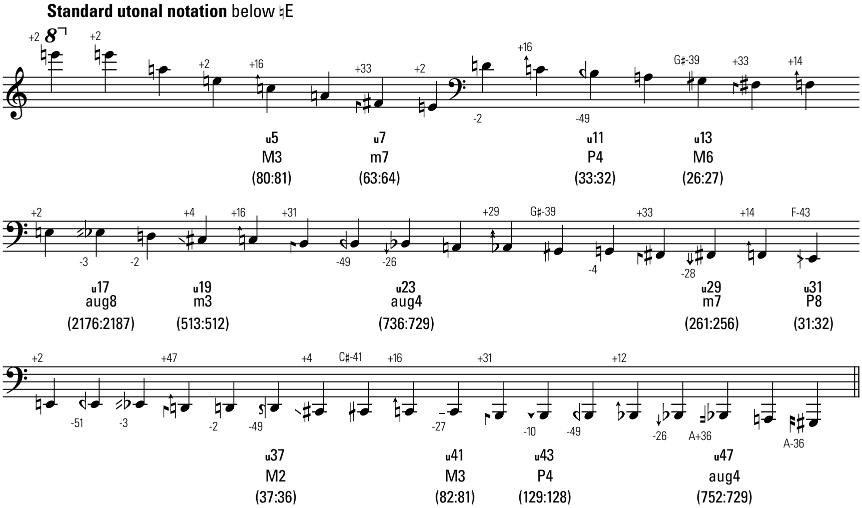
building upon the original HEJI devised by Marc Sabat and Wolfgang von Schweinitz

Ratios represent the amount of modification of the Pythagorean notes by each additional symbol,
cents indications are deviations that would be shown on a tuning meter with A = 0 cents

Standard otonal notation above ♮A



Standard utonal notation below ♮E



The “spiral model”

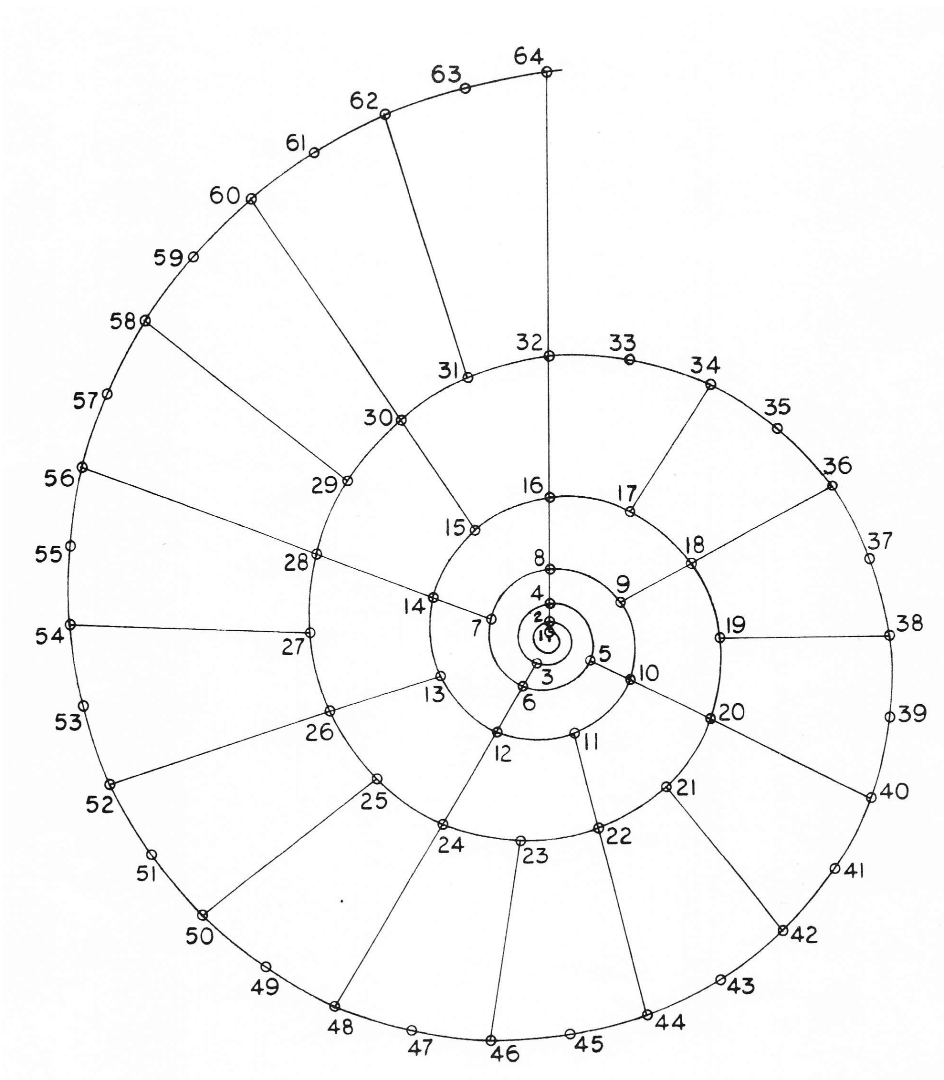


Figure 2: “The Harmonic Series as a Logarithmic Spiral” by Erv Wilson (1965)

Pitches generated by harmonic and subharmonic series evolve out of the composition of numbers as unique products of primes. Each power of 2 is equivalent to transposition by one octave, which is usually considered an equivalence relation. All octaves of a frequency belong to the same pitch class, and all powers of 2 are considered musically equivalent in some sense. It is, therefore, common to express a pitch class as a ratio in *normalised* form, by adjusting the power of 2 accordingly to obtain a fraction between $\frac{1}{1}$ and $\frac{2}{1}$, a conceptualisation championed by Partch.

As the harmonic series progresses, octave by octave, each of the previously established intervals recurs and is divided. The first octave, 1 : 2, is simply the undivided interval $\frac{2}{1}$; the second octave, 2 : 4, is a $\frac{2}{1}$ divided by 3° into two parts, 2 : 3 : 4, producing the intervals $\frac{3}{2}$ and $\frac{4}{3}$; the third octave, 4 : 6 : 8 is divided by 5° and 7° to produce 4 intervals, etc. This method of dividing intervals always generates epimoric steps because they are frequency ratios of successive harmonics and thus near-equal, "harmonious" small-number divisions. One well-known example occurs in the fourth octave of the series, where the major third 4 : 5 is divided 8 : 9 : 10 to produce two nearly equal whole tones, as discussed above in terms of the Euler Lattice.

Each odd partial introduces a new pitch class, and with it new, specific, and typical JI sonorities, while every even number is a repetition of an earlier partial transposed by an octave. Since the primes by definition are indivisible, their pitch classes may only be reached in a single step from the reference frequency. If the interval subtended cannot be tuned, then that note remains, in some sense, harmonically inaccessible; it may only be established by measurement or reason, but not directly by perception. This boundary is often expressed by imposing a "prime-limit" on the ratios included in an harmonic space.

On the other hand, composite numbers that are odd play a special role: by *combining* two or more odd factors, these partials represent bridges between pitch classes. Composite numbers may be broken down into smaller factors, and these, when odd, produce different pitch classes. As long as the primes involved are all salient, the composite interval may be established in multiple ways by a finite sequence of steps. For example 9° is the 3° of the 3°, or 15° may be taken in two ways, as the 3° of 5° or the 5° of 3°.

In her article "Conceptualising the Spiral: Humans Intersecting with Harmonic Space" (p. 250 of this volume), composer Catherine Lamb reflects on her ongoing work with a model of the harmonic series as a logarithmic spiral. She has developed a personal version of Erv Wilson's diagram from 1965,[12] visualising the cyclic division of each octave by odd partials, as described above. In general, the entire range of frequencies may be expressed as ratios from a single reference, e.g. $\frac{1}{1} = 440$ Hz. The range of these possible ratios is the set of all positive real number values $r > 0$. Since frequency relations are

12 Erv Wilson, "The Harmonic Series as a Logarithmic Spiral", 1965, http://www.anaphoria.com/harm-subharm.pdf.

multiplicative, the expression *$log_2 r$* allows intervals to be measured additively in terms of the base, in this case 2, which simply calculates the number of octaves comprising an interval. By linking *$log_2 r$* with a rotational angle θ such that one complete rotation through 360° = 2π radians is scaled to represent one octave, Wilson devised a useful visualisation of the harmonic series and its cyclic generation of octave-equivalent pitch classes.

The polar equation $log_2 r = \frac{\theta}{2\pi}$ generates a spiral upon which every positive real ratio r projects onto a specific angle θ along with all of its octave transpositions, which lie on a straight line or "spoke" radiating from the origin. Thus, the angle represents a pitch class, with a unique value $2\pi log_2 r$. Ellis's well-known measure of interval size using $\frac{1}{1200}$'s of an octave defines the cents of a ratio r as $cents = 1200 log_2 r$, so $\theta = \frac{\pi}{600} cents$. The harmonic series is a special subset of r, namely, that consisting of all of the natural numbers n. Since the spiral is self-similar at all scales, curling inward infinitely towards the origin even beyond a radius of 1 (the generator), the subharmonic series is depicted along the same curve by all of the reciprocals of natural numbers $r = \frac{1}{n}$. The portion of the spiral between $r = 1$ and $r = 2$ represents the normalised octave of pitch classes. It is interesting to note how Wilson's spiral curves *clockwise*, implying the use of only negative angles.[13] In examining the logarithmic spiral in as simple a form as possible, the remaining spirals in this article curve *counterclockwise*, thereby graphing positive angles.

This visualisation is particularly useful when examining the distribution of various pitch-class subsets. Each fraction occurs at some unique point on the spiral curve, and by projecting a line from the origin through this point, there will always be one unique point on the normalised segment of the spiral that marks the intersection of the line and the spiral. As fractions are added to a pitch set, their normalised form may be visualised by projecting "spokes" onto the normalised segment and observing the distribution of points (as in an analysis of the 17-limit tuning of the blues scales in Figure 3). Since each octave subtends a full rotation around 2π radians, this model allows the symmetries and distribution of intervals in various pitch sets to be easily conceptualised and compared. In this sense, it offers a window into how one might approach imagining the multidimensional character of harmonic space as it is extended beyond the triadic generators 2°, 3°, 5°.

[13] In polar coordinates, the angle θ is conventionally measured from the horizontal axis extending rightward as 0, thus counterclockwise rotation generates increasing positive values.

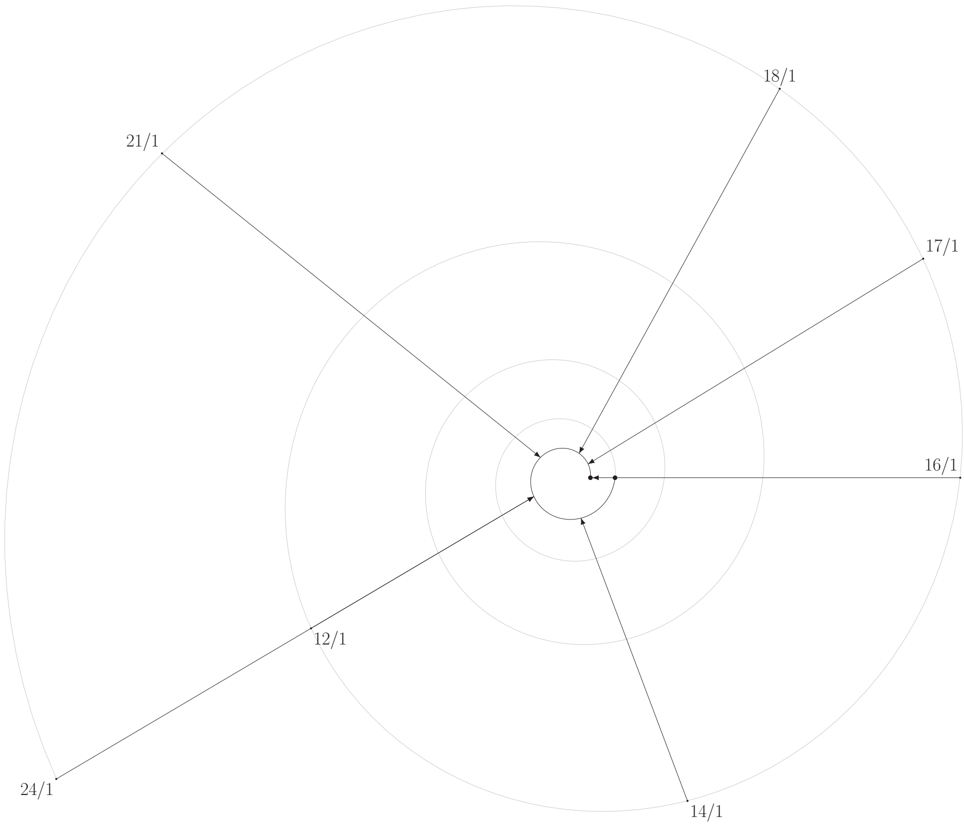


Figure 3: The 17-limit blues scale, tuned as harmonic series segment 12:14:16:17:18:21:24, visualised on the logarithmic spiral. These harmonics, expressed as fractions over 1, may be projected inward to the black segment between $\frac{1}{1}$ and $\frac{2}{1}$ (denoted by bold dots) to map the scale as pitch classes with respect to the original fundamental, namely, $\frac{1}{1}$, $\frac{17}{16}$, $\frac{9}{8}$, $\frac{21}{16}$, $\frac{3}{2}$, and $\frac{7}{4}$.

James Tenney's harmonic space

"It follows naturally that the apprehensions of the senses are determined and bounded by those of reason, first submitting to them the distinctions that they have grasped in rough outline – at least in the case of the things that can be detected through sensation – and being guided by them towards distinctions that are accurate and accepted. This is because it is a feature of reason that it is simple and unmixed, and is therefore autonomous and ordered, and is always the same in relation to the same things, while perception is always involved with multifariously mixed and changeable matter, so that because of the instability of this matter, neither the perception of all people, nor even that of the very same people, remains the same when directed repeatedly to objects in the same condition; but it needs, as it were as a crutch, the additional teaching of reason."

— *Claudius Ptolemy*[14]

Arguably, one limitation of the spiral is a direct consequence of its 2-dimensionality: it only allows for two quantities to be encoded when representing a pitch. In the shift from Cartesian to polar coordinates, the information pair (numerator, denominator) is effectively transformed into the pair (ratio, pitch-class height), which suggests a clear geometric and musical conceptualisation. As notes are added, the circle of pitch classes filling up the octave 1 : 2 becomes more densely populated.

To better discern and differentiate harmonic relationships, it is useful to return to a lattice model, extending Euler's 2-dimensional tone network (discussed above). Recall that every natural number has a unique prime factorisation, which may be mathematically written as a product of primes with positive exponents (primes that are not factors receive the exponent 0). Every fraction is simply a division of two such products, which may be simplified to a product of primes with *integer* exponents, some of which may be positive, thus corresponding to the factors of the numerator, and some negative, corresponding to the factors of the denominator.

In "John Cage and the Theory of Harmony",[15] James Tenney describes what he calls "harmonic space": a lattice of infinite dimensionality in which every axis corresponds to a prime number p_i. In each of these prime dimensions, units of length $\log_2 p_i$ along the axis enumerate the exponent of p_i in the prime factorisation of a given rational number. Thus, each lattice point represents a unique rational number and each rational number is represented by a unique point, corresponding to a frequency ratio measured from some

14 Andrew Barker, *Greek Musical Writings, Volume II: Harmonic and Acoustic Theory* (Cambridge: Cambridge University Press, 1997), 276–277.

15 James Tenney, *From Scratch: Writings in Music Theory*, ed. Larry Polansky et al. (Champaign-Urbana: University of Illinois Press, 2015), 280–304.

reference. The ratio itself may be concisely represented as "harmonic space coordinates" in the form of a so-called *monzo*, which is a vector comprised of the exponents of the primes involved. For example, the ratio $\frac{35}{18}$ has a prime factorisation of $2^{-1}3^{-2}5^{1}7^{1}$, so its corresponding monzo would be expressed as $|\ {-1}\ {-2}\ 1\ 1 >$. Typically, the length of the monzo is determined by the prime-limit of the interval it describes.[16]

Just as was observed in the Euler diagram, moves in this lattice (i.e. *vectors*) describe intervals, and combinations of vectors (paths, shapes) describe chords or aggregates. In harmonic space a vector may be describing a movement through several dimensions, but it is easily grasped algebraically in terms of the monzos. To calculate an interval between two lattice points, their coordinates are simply *subtracted term-by-term* (i.e. 3 steps descending in the 2-dimension, 1 step ascending in each of 3- and 5-dimensions, would imply the interval $\frac{15}{8}$, and this is the *vector difference* between the coordinates of $\frac{5}{2}$ and $\frac{4}{3}$). Pitch-class equivalency is represented by extending a line through a point, parallel to the 2-axis; all lattice points falling on this line would be various octave transpositions. Algebraically, this is the same as ignoring the exponent of 2, allowing it to range arbitrarily. Pitch height (in the spiral this was the angle to the origin) may also be given a geometrical interpretation by constructing a line at $45° = \frac{\pi}{4}$ with respect to all of the prime axes, and then projecting orthogonally onto this line. It might be imagined as a vertical line around which each of the prime axes are rotated.[17] Then the normalised octave would be the pitch-height line segment between 0 and 1, and all lattice points project lines onto unique points within this bound.

As interlocking harmonic and subharmonic series extend out from a central generator, each new note becomes a potential generator of new series or, even more generally, it becomes a possible harmonic or subharmonic partial of infinitely many series. This kind of endless self-similarity benefits from a geometrical model that is also similarly structured at every point while extending infinitely in all directions.

One especially useful consequence of this model is that it becomes possible to readily conceptualise subsets of harmonic space spanned by an arbitrary subset of the prime axes: for example, the tuning of La Monte Young's *The Well-Tuned Piano* may be described by reducing to the axes 3 and 7. Since octaves are considered equivalent in his tuning, the 2-axis values may be collapsed or projected to an exponent value 0 (i.e. eliminating all non-zero powers of 2).

[16] Zeros denote intervening primes that are not part of the factorisation (e.g. $|\ {-2}\ 0\ 0\ 1\rangle$ for $\frac{7}{4} = 2^{-2}7^{1}$). Note that the use of ket notation has become standard in recent times, among other reasons to facilitate calculations of errors produced in various systems of temperament when approximating just intervals. For more information, see https://en.xen.wiki/w/Mike%27s_Lecture_on_Vector_Spaces_and_Dual_Spaces.

[17] Note that the pitch-height line passes *through* the n-dimensional space, it does not *add* a new dimension. In 2 dimensions it would be the line $x = y$, and in n dimensions it is the line $x_1 = \ldots = x_n$. By the Pythagorean theorem, the p_ith exponent a_i is projected onto the height $\frac{1}{\sqrt{2}} a_i \log_2 p_i$, thus the lattice point R is projected onto $h(R) = \frac{1}{\sqrt{2}} \log_2 R$ and $cents(R) = 1200\sqrt{2}h(r)$.

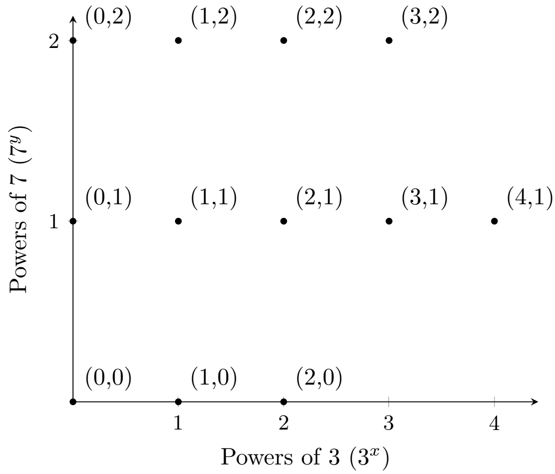


Figure 4: A (3.7)-subgroup lattice depiction of *The Well-Tuned Piano* tuning.

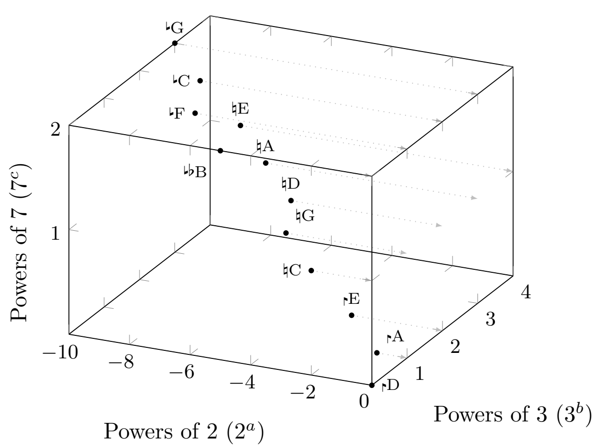


Figure 5: A three-dimensional harmonic space subset mapping the ratios of *The Well-Tuned Piano* tuning.

While mathematically defined harmonic space may extend indefinitely, the capacity to traverse it by vectors that represent salient singular harmonic “steps” is, in fact, bounded. The geometry of harmonic space is constrained by the mixed and inconsistent nature of sensation, as Ptolemy so clearly articulates in the text cited above. But what bounds might reasonably reflect perception, and how?

Harmonic distance and psychoacoustic tuneability

After first devising the HEJI notation, in 2005 Sabat compiled a list of all ratios found between pairs of the first 28 partials from an harmonic series, organised by increasing pitch height. Each interval was tested in combinations of various acoustic and electronic timbres to determine to what degree the intervals might be considered "tuneable" by ear.[18] As might reasonably be expected, the simpler sounds to distinguish were those ratios with smaller numbers.

There is, however, a simple algorithmic method by which progressively more "dissonant", less immediately "salient" ratios may be deduced. This follows a procedure known in mathematics as the *Farey Sequence*, described in an earlier article by the present authors. First, a definition: the *mediant* of two fractions is the sum of both numerators divided by the sum of both denominators. It can be shown mathematically that a mediant always lies between both parent fractions, and if they are in lowest terms, then their mediant is also. Beginning with $\frac{0}{1}$ and $\frac{1}{1}$, successively interpolated mediants obtain an ordering of the rational numbers between 0 and 1. Each number is produced in lowest terms and occurs at a singular point in the process. The mediant of $\frac{0}{1}$ and $\frac{1}{1}$ is $\frac{1}{2}$; the mediant of $\frac{0}{1}$ and $\frac{1}{2}$ is $\frac{1}{3}$, the mediant of $\frac{1}{2}$ and $\frac{1}{1}$ is $\frac{2}{3}$; and so on. Taking all the mediants whose denominators do not exceed a given value *n* produces the Farey Sequence of order *n*. Two neighbouring fractions in a Farey Sequence are called a Farey pair. For any two musical intervals forming a Farey pair, the most consonant ratio lying between them is their mediant. For example, $\frac{5}{4}$ and $\frac{4}{3}$ are a Farey pair. The next most consonant interval between the major third and the perfect fourth is the very large septimal major third with ratio $\frac{9}{7}$.

Informal testing suggests that under good circumstances tuneability may be extended up to (at least) 23°. The ratio $\frac{23}{12}$, which is somewhat wider than a Pythagorean major seventh, demonstrates a small but better-than-chance degree of tuneability. At the same time, many other similarly "large" ratios within these numerical bounds, like $\frac{17}{13}$, may not so easily be distinguished or accurately tuned. So what are the parameters constraining tuneability, other factors (timbre, register, etc.) being equal? What might be a reasonable way of drawing on these sounds to construct and bound a set of rational pitch classes, so that the resulting harmonic space is readily traversable by intervals that may be perceived, without resorting to deliberate mistuning (temperament), while offering a free and fluid experience of musical harmony?

Tenney defined a measure of harmonic space called *harmonic distance* (HD), which is proportional to the product of the numerator and denominator of each ratio. For a ratio

18 This is a term that is based on a complex, empirically evaluated interaction of multiple factors. These include: a similar volume and sufficiently stable sustain; perceiving beating between partials contained within the balance of combined timbres; observing the intonation, resonance, and beating of combination tones; listening for periodicity manifested in the phase of superposed partials. It is obviously predicated on individual experience and the particularities of specific acoustic spaces.

$\frac{a}{b}$ in lowest terms, $HD = log_2(ab)$. It is interesting to note that ab alone, which calculates the lowest common partial shared by the respective harmonic spectra above a and b,[19] was proposed in the 16th century by Benedetti as an effective measure of the relative dissonance of a ratio.[20] HD of an aggregate $a_1{:}\ldots{:}a_n$ would be log_2 applied to the least common multiple of the a_i.

By establishing a frequency ratio, two pitches combine as though they were two partials of an unsounded virtual fundamental, their *periodicity pitch* F_p. The periodicity of the combined waveform of a and b is in many circumstances actually perceived. Sometimes it sounds like a pitch, and sometimes it is simply a rhythmic pattern of pulsation. Sometimes the first-order difference tone $|a - b|$ is the same frequency as F_p; sometimes it is a higher partial of F_p, reinforcing it. Each of a and b may produce an harmonic spectrum, and these partials are also partials of F_p; ab is their first partial unison.

Observe that HD is by definition a symmetric measure ideally suitable to intervals, since these are identical when considered either upward and downward. For aggregates, however, reversing the order of successive intervals (i.e. major and minor triad) *does* have a definite effect on perceived harmonicity, because this depends also on the psychoacoustic property of overtonal fusion, but the HD does not change. Tenney therefore suggested one other quantitative measurement of psychoacoustic harmonicity, called *intersection*,[21] evaluated by the expression $I = \frac{a + b - 1}{ab}$. The smaller the pitch distance F_p from to ab and the more fully the spectrum of F_p is sounded, the stronger the perceived sensation of consonance. Intersection may also be generalised to aggregates by counting the unique partials of each component pitch up to their least common multiple.

Since $log_2(ab) = log_2(a) + log_2(b)$, another interpretation of HD is possible. The common partial of the fraction $\frac{a}{b}$ is broken down into two "jumps", from 1 to a (overtonal) and from 1 to b (undertonal). The respective absolute-value pitch distances in octaves from fundamental to partials a and b, respectively, are simply added, producing the pitch distance to ab. Whether the interval in question is itself constructed upward or downward from a generating frequency is ignored. By virtue of the unique prime factorisation of each number, a and b may be (potentially) further broken down into *elementary steps* $1 : p_i$ for each prime factor dividing them. Thus, measuring HD is equivalent to the sum of all absolute distances from the fundamental to a ratio's constituent primes. It evaluates the harmonicity sum of the totality of steps needed to construct an interval, and does not care in which order or direction those steps are taken.

Observed as a sum of elementary steps, it becomes clear that HD ignores the real-world fact that as primes get larger – in particular as they exceed the ratio $\frac{9}{1}$, namely, for

19 *ab* is the *b*th partial of *a* and the *a*th partial of *b*.

20 Giovanni Battista Benedetti, *Diversarum Speculationum Mathematicarum et Physicorum Liber* (Turin, 1585), 277–283.

21 Tenney, *From Scratch*, 248.

prime partials beyond 11° – their associated elementary steps become too wide to support psychoacoustically salient harmonic relationships or consonances. To tune these higher primes, which practice demonstrates to be *possible*, factors of 2 need to be introduced in the denominator to keep the interval within 3 octaves and allow its consonance to be directly grasped. For example, $\frac{11}{1}$ is difficult to establish, but $\frac{11}{2}$ is clear and consonant. This is similarly true for 13°.

Therefore, we propose a psychoacoustically adjusted form of HD as follows. By Tenney's definition, for a ratio expressed as

$$\prod_i p_i^{a_i},$$

$$HD = \sum_i |a_i| log_2(p_i)$$

Instead, for prime factors above 7°, let the scaling factor $log_2\left(\frac{p_i}{9}\right)$ be introduced so

$$HD_a = \sum_i^4 |a_i| log_2(p_i) + \sum_{i>4} |a_i| log_2\left(\frac{p_i^2}{9}\right)$$

Here the first 4 primes, 2, 3, 5, 7, which lie within the $\frac{9}{1}$ are treated conventionally, while primes 11 and higher gradually are scaled so that each additional octave beyond the $\frac{9}{1}$ introduces a factor of 2.

When generating sets of pitches, it is often useful to establish the same pitch classes in all octaves. Not all such relationships may be tuneable in closed position, but there is often some wider voicing of an interval that *is*, allowing the pitch class to be tuned in a different octave and then transposed back into normalised form. For example, consider the ratio $\frac{16}{11}$, which is relatively dissonant and difficult to tune in closed position. The ratio $\frac{2}{11}$, on the other hand, is much more readily perceived as consonant. This procedure suggests how the adjusted HD might be extended to evaluate relative tuneability in a set of pitch classes.

Given a normalised ratio $\frac{a}{b}$:

– examine all possible octave transpositions of the two notes, *which do not exceed a total transposition of* $\frac{8}{1}$ (i.e. adjust the 2-exponent);[22]
– find the optimally tuneable ratio with minimum adjusted HD (i.e. minimise 2-exponent absolute value);
– calculate the adjusted HD of this ratio and assign this value to be the optimised pitch-class HD. For example, in the example suggested above, the upper note of $\frac{16}{11}$ is transposed down by 3 octaves to produce the ratio $\frac{2}{11}$, which is the pitch class ratio's optimal voicing. The 2-exponent is reduced from 4 to 1. Therefore, the adjusted pitch-class HD of $\frac{16}{11}$ is measured as $\frac{2}{11}$, increasing its evaluated saliency.

Part 2: 933-Tone-Tree Harmonic Space

Conditions for an enharmonically viable subset of harmonic space

Assuming that sounds intended to articulate salient relations in harmonic space should remain for the most part true and undistorted ("pure"), what are possible options for defining musically flexible subsets of pitches? In particular, how is it possible to do so without resorting to deliberate mistuning (temperament)? The perception of simultaneous sounds (chords) favours relationships of smaller numbers and intervals outside the critical band (ca. $\frac{7}{6}$ and greater), and small deviations from these ratios are recognised with a high degree of sensitivity. Substantial variations in size between combinations of these "pure-tuned" ratios produce an infinitude of noticeably *microtonal* differences, which may easily produce a sense of "out-of-tuneness". On the other hand, perception of successive sounds (melodies) favours relationships of larger numbers and intervals within the critical band (ca. $\frac{8}{7}$ and smaller), preferring evenness and fine variation in interval size to suggest clarity and purity. Each paradigm supports a different and potentially opposed continuum of simplicity/complexity.

Harmony in its musical sense, namely, as discernible progressions, sequences, and relations of pitched sonorities both simultaneous and successive, is therefore animated by a tension between locally compact[23] pitch aggregates in harmonic space, psycho-

[22] This is equivalent to assuming that octave transpositions of notes by up to 3 octaves (by $\frac{2}{1}$, $\frac{4}{1}$, or $\frac{8}{1}$) do not effectively add any HD, because in practice this process is easily accomplished by ear. The mathematical process involved reduces the absolute value of the 2-exponent of the ratio by at most 3.

[23] Compactness is used in this paper somewhat informally. It refers to the process of minimising harmonic distance within a set of pitches. For example, the lattice of tones defining the Ptolemaic major scale consists of neighbouring points along the 3- and 5-axes of harmonic space, projected into the same octave by transposition along the 2-axis. It is in some sense compact: intervals available from each note to most other ones in the scale are relatively consonant *because the pitches are nearby*. Developing a more rigorous discussion of this topic, including comparison of various measures and bounds and considering how these affect melody and chord aggregates is an active area of ongoing research.

acoustically plausible enharmonic transitions ("substitutions"), which momentarily privilege pitch-height distance (proximity) over harmonic distances within the lattice, and a practical wish to maintain some degree of pitch consistency or "centring".

Without enharmonic transitions, one would have the kind of traversal that Lou Harrison described as "free style", relating pitches to other pitches locally by salient harmonic relations, and allowing the point of reference to drift from moment to moment. Such an approach, though common, may very quickly move away from a pragmatic JI notation, because there is no fixed backbone of pitches to refer to, only relative intervals. Composers like Toby Twining, in his vocal work *Chrysalid Requiem*, have devised solutions where accumulations of accidentals representing multiple septimal intervals, for example, are enharmonically renotated simply by temporarily shifting the reference frequency. However, this method may be too disorienting when attempting a practical correlation to most instruments other than the completely untethered human voice or theremin, etc. It is interesting to consider compositional approaches that disguise or delineate awareness of drifting pitch and how these may affect formal relationships in a piece.

Alternatively, one might restrict oneself to a kind of "strict style", an extension of modal or sruti-based systems, in which a particular region of harmonic space is *strictly* mapped out by the music exactly as it allows (i.e. La Monte Young's *The Well-Tuned Piano* or Wolfgang von Schweinitz's *Plainsound Glissando Modulation*). Such an approach is compatible with JI notation and with a limited sense of tonal "modulation", while demanding that harmonic steps taken be eventually retraced, perhaps in a different order, but nonetheless returning home before proceeding in a new direction.

Recall the three ♭B's in the extension of Euler's lattice discussed above. In this case, each of the three enharmonic differences between them is large enough to change the sense in which an interval is interpreted. With certain constellations of notes it might suggest a move between two *different* consonances, but more likely it would simply shift a sonority from consonance to dissonance, producing a "wolf" interval. To compose in quasi-"free" style, more subtle degrees of substitution would be necessary. For example, in al-Farabi's tuning of the oud, which was a modification and extension of al-Kindi's, there are 5 strings tuned in Pythagorean perfect fourths (ascending $\frac{4}{3}$ ratios) with frets at $\frac{8}{9}$ths and $\frac{27}{32}$nds of the string length (see Figure 6). On the same string, the difference between these two frets is a limma 243 : 256. Two strings over, the difference is an octave less an apotome, and three strings over, the interval produced is 6561 : 16384, which differs from the simple ratio 2 : 5 by only 32768 : 32805 (less than 2 cents). This well-known "schisma" is perhaps the first of several "near-enharmonic" proximities to come into common use and is the difference between 8 descending Pythagorean perfect fifths (normalised to a diminished fourth) and one ascending $\frac{5}{4}$.

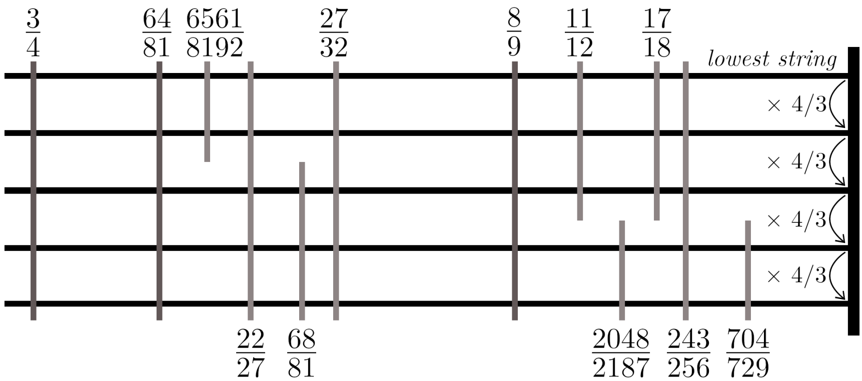


Figure 6: Al-Farabi's divisions of the strings of the oud.

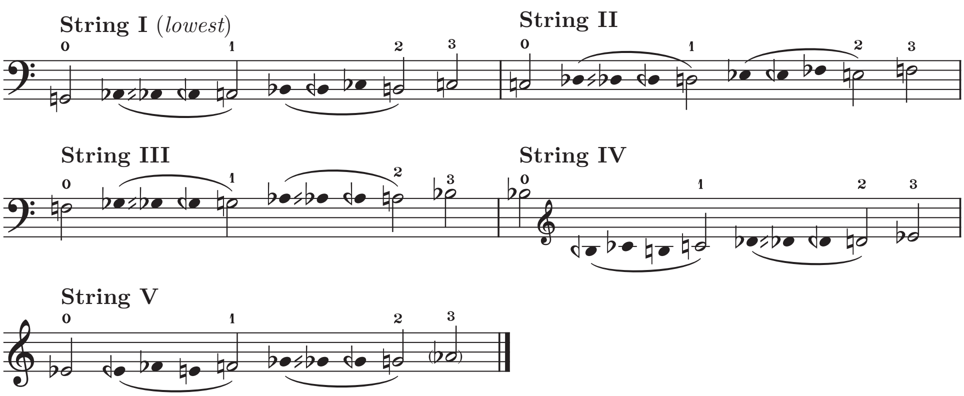


Figure 7: Al-Farabi's divisions transcribed as rising tones in HEJI notation with string I tuned to G.

The probabilistic evaluation of "harmonic entropy" proposed by Paul Erlich quantifies the question, in his own words, "how confused is my brain when it hears an interval" as it "attempts to fit the stimulus to an harmonic series".[24] Certain large-number ratios in the near vicinity of better-known simple ratios have a higher probability of being perceived as mistunings of the simpler ratios rather than eliciting distinct "sonic fingerprints" in their own right. This characteristic of harmonic auditory cognition allows for a greater harmonic

[24] Paul Erlich, "Harmonic Entropy", 2004, http://www.soundofindia.com/showarticle.asp?in_article_id=1905806937.

variety to be inferred from a reduced set of pitches (in this case, as defined by the positions of the frets described by al-Kindi), namely, Ptolemaic ratios combining primes 2, 3, and 5 may be represented by means of the extended series of $\frac{3}{2}$ perfect fifths. Note that the natural tolerances of an acoustic instrument (bow, breath, finger pressure, etc.), even in a highly controlled context, vary within a comparable range of ca. 2 cents.

In 12-tone Pythagorean tuning, which was used as one of the first systems applied to organs, the sequence of 12 perfect fifths obtains four diminished fourths that are each one schisma *smaller* than $\frac{5}{4}$'s. The observation of this near-enharmonic proximity led to various transitional just tunings[25] before meantone temperaments became common practice. In several historical tunings, these diminished fourths were "corrected" and tuned as pure $\frac{5}{4}$'s, producing a perfect fifth narrowed by a schisma, ca. 700 cents, at the point of transition. Note that the perfect fifth on a modern equal-tempered piano also differs from a pure $\frac{3}{2}$ by approximately one schisma; arguably, the perceived "in-tuneness" of 12-EDO lies in its close proximity to Pythagorean tuning, rather than in its ability to (poorly) represent Ptolemaic ratios or any of the higher primes.

Helmholtz cites Arabic and ancient Persian sources for their remarkable implementation of the schisma, and describes a "schismatic" organ tuning he put into practice that unites the Pythagorean and Ptolemaic systems in a single, enharmonic framework of 24 pitches. Observe, however, that 53 tones are actually necessary to produce an enharmonically viable simulation of freestyle in the 5-limit, enabling undistorted representation of its characteristic intervals, such as $\frac{3}{2}$, $\frac{5}{4}$, and $\frac{81}{80}$. $\frac{1}{53}$ of an octave closely represents both the syntonic and the Pythagorean commas, which differ by a schisma; after 53 untempered $\frac{3}{2}$'s the deviation from the starting pitch is only 3.6c. By the 1870s, von Oettingen had conceived of a 53-tone justly tuned keyboard instrument called the "orthotonophonium", with a 53-tone JI tuning centred around D, taking advantage of schisma-altered "near-fifths" to emulate a continuous Tonnetz. Oettingen begins by adding 4 ascending and descending $\frac{3}{2}$'s on either side of D to produce a chain of nine perfect fifths. Taking $\frac{5}{4}$'s above and below eight of these fifths accordingly, he extends the chain of fifths with an enharmonic schisma alteration, and continues similarly until the system generates 26 "fifths" on either side of D.

[25] Klaus Lang, *Auf Wohlklangswellen durch der Töne Meer: Temperaturen und Stimmungen Zwischen dem 11. und 19. Jahrhundert* (Graz: Institut für Elektronische Musik (IEM) an der Universität für Musik und darstellende Kunst in Graz, 1999).

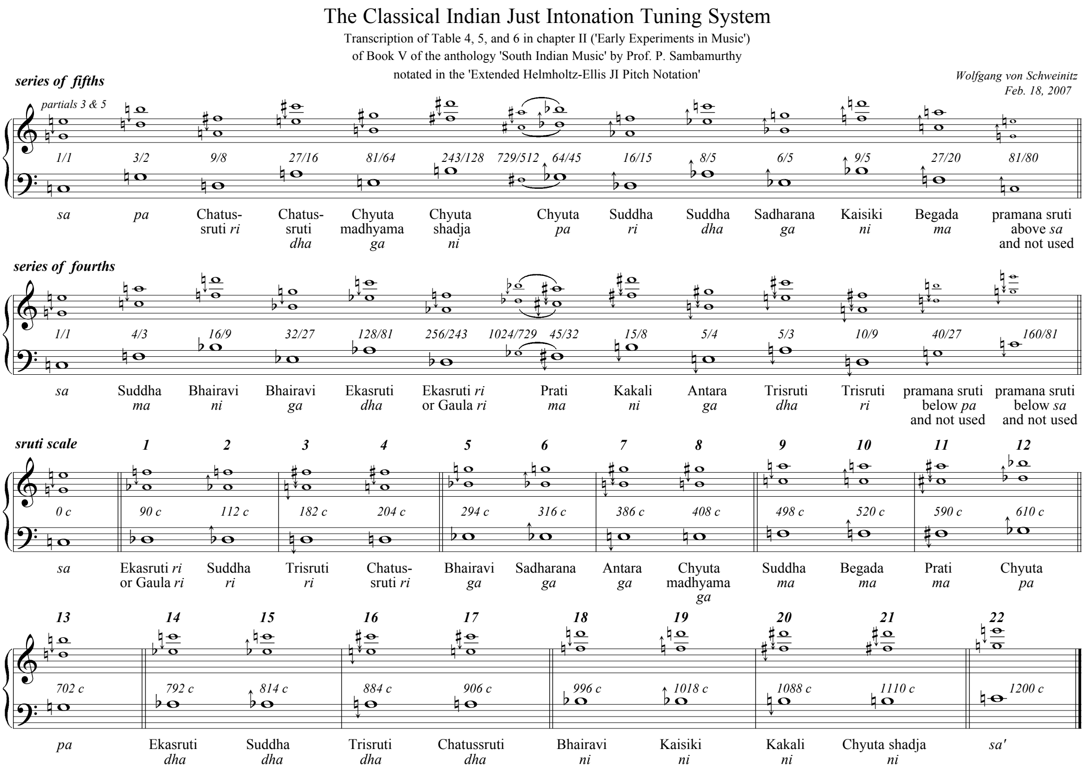


Figure 8: The Indian sruti system according to Sambamurthy, transcribed by Wolfgang von Schweinitz.[26]

In his speculative reconstruction (see Figure 8) of the Indian sruti system based on rational intervals, which were apparently not used in early Indian music theory, Pichu Sambamurthy introduces a substitution by one schisma after six ascending and descending $\frac{3}{2}$'s from “sa”, in a manner conceptually akin to the tuning implemented by Oettingen. He then continues the respective Pythagorean chains from these new notes, stopping before reaching syntonic-comma-altered enharmonics of the initial sa or pa. This produces a total of 24 notes, 12 pairs of nearby tones, two pairs of which deviate by one schisma, resulting in an effective pitch set of 22 srutis. This example shows how a limited “strict style”, continued far enough, might approach a perceptually near-perfect “free style” in a higher limit. Musicians practising Indian traditions like Dhrupad, in which very precise

26 Wolfgang von Schweinitz, “The Classical Indian Just Intonation System”, 2007, https://www.plainsound.org/pdfs/srutis.pdf.

differences of tuning are recognised, often resist attempts to quantify exact tuning of the srutis. In addition to well-known 5-limit intervals, there are many other subtle enharmonic variations, which come into play in actual music-making when following the melodic logic of specific ragas against the drone. An actual enumeration of commonly played srutis might more realistically number in the hundreds and likely would involve primes beyond 5.

It is interesting to note Oettingen's advocacy of a just tuning rather than an arguably even more "perfect" sounding micro-temperament, in which the schisma is "imperceptibly" distributed across 8 slightly-tempered fifths. Since such a subtle mistuning falls below the threshold of stability for most acoustically generated tones, it might be considered by some a more "ideal" compromise solution for musical practice, compatible with *adaptive just intonation*. An argument against such an approach is the fact that, unlike extensions of the JI pitch space, micro-temperaments require frequencies of JI ratios to be subtly adjusted with each new vertical structure. However, these *intentional mistunings* can only be realised accurately with fixed-pitch instruments or by relying on digital tuners.[27] On the other hand, in JI, even with many notes, there is always a possibility that pitches may be *exactly tuneable* on acoustic instruments by ear alone. This perhaps utopian yet imaginable distinction – of composing harmonic space in accordance with perception – compels us to favour experimental explorations of this tension between the measurable and the sensible.

At this point, it is possible to list, in a loosely prioritised order, some desirable conditions for enharmonically viable subsets of harmonic space. These might serve as suitable sets of pitches used by a computer-controlled fixed-pitch instrument or as relatively "complete" depictions of the extent of harmonic auditory cognition.

- A 47-limit HEJI notation that does not exceed 3 accidentals to facilitate real-time music-making.
- Allowing enharmonic substitutions within approximately one schisma at each point of the pitch set so that salient ratios at harmonic boundaries of the system may effectively be emulated.
- A relatively even distribution of pitch classes within the octave and of melodic steps within individual harmonic and subharmonic series.
- An excellent representation of the extended Pythagorean and 5-limit without wolf intervals (false fifths or false thirds) unless desired compositionally.
- Tuneable intervals through the 23-limit from multiple points of reference.
- A set of tonalities: otonalities and utonalities on all of the chromatic notes.
- A systematic search ordering, allowing any arbitrary frequency to be mapped to the harmonically simplest pitch class available within a desired tolerance. This would enable a computer instrument to interpret frequencies as ratios in a manner similar to a human listener.

[27] Note the prevalence of pitch correction software in music production today, so that musicians may actually "play" or "sing" in 12-tone equal temperament.

The Stern-Brocot Tree: Generating a compact harmonic space

In January 2020, we published a paper investigating properties of Farey sequences applied to the distribution of harmonic nodes on stringed instruments, as well as the ordering of fractions, representing pitches or intervals, that may define an harmonic space (briefly discussed above in relation to harmonic distance and tuneability). Note that the Farey sequence order $n + 1$ does not contain *all* of the possible mediants in the Farey sequence order n; only those mediants with denominators $\leq n + 1$ are included, and the fractions are always limited to the range between 0 and 1.

A related algorithm – the *Stern-Brocot Tree* – is an infinite binary search tree that eventually lists every positive rational number exactly once and always in lowest terms. Each new row of the tree is populated by all the mediants of neighbouring fractions from the union of all previous levels of the tree. Beginning, conceptually, with the pseudo-fractions $\frac{0}{1}$ and $\frac{1}{0}$, which are always excluded from the tree itself, the Stern-Brocot Tree order 1 is defined as their mediant: $\frac{1}{1}$. This becomes the parent of two children, the mediants of $\frac{1}{1}$ with $\frac{0}{1}$ (i.e. $\frac{1}{2}$) and with $\frac{1}{0}$ (i.e. $\frac{2}{1}$). All three fractions ($\frac{1}{2}$, $\frac{1}{1}$, $\frac{2}{1}$) form the Stern-Brocot Tree order 2. The next iteration of this procedure is shown in Figure 9.

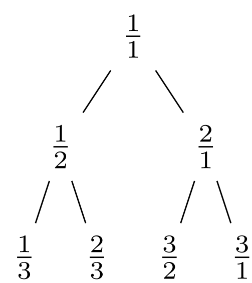


Figure 9: The Stern-Brocot Tree order 3.

The number of fractions in a given row n is always 2^{n-1}. By collapsing the tree, one generates a sequence of 2^{n-1} ordered fractions in lowest terms, bounded by the fractions $\frac{1}{n}$ and $\frac{n}{1}$. This sequence is symmetric around $\frac{1}{1}$, containing pairs of reciprocals, suggestive of the way harmonic and subharmonic series emerge around a given reference, or, more simply, how *any interval* may be generated equally feasibly *above* or *below* any given pitch ($\frac{1}{1}$). It is exactly this symmetry that makes the Stern-Brocot Tree an ideal generator for notes comprising a pitch space.

The Stern-Brocot sequence order 9 (SB-9) with 511 fractions – plotted on a logarithmic spiral in Figure 10 – is the largest order sequence within the 47-limit, i.e. producing ratios that are all notatable in HEJI. A translation into notation with ♮D as $\frac{1}{1}$ verifies that no more than 3 accidentals are needed to notate these pitches. Additionally, as the Stern-Brocot Tree grows, ratios between neighbouring fractions in the sequence approach $\frac{1}{1}$. Musically, this manifests as a melodic step between neighbouring notes approaching *unison*. SB-9 is the first sequence in which steps smaller than the schisma appear, allowing for almost imperceptible enharmonic substitutions. These properties satisfy the first two conditions established in the preceding section for an enharmonically viable extended JI pitch space.

Regarding the question of evenness, one faces a similar dilemma to that encountered by Partch when he collated the pitch set produced by his 11-limit tonality diamond. In the case of SB-9, gaps between fractions range from as large as a whole tone ($8{:}9 \approx 204$c, found between $\frac{8}{1}$ and $\frac{9}{1}$ and their reciprocals) to somewhat smaller than a schisma ($1155:1156 \approx 1.5$c, found between $\frac{34}{21}$ and $\frac{55}{34}$ and their reciprocals). A simple procedure to begin filling in these gaps is to normalise the sequence. 129 of the fractions in SB-9 already lie in the range between $\frac{1}{1}$ and $\frac{2}{1}$. After normalising the remaining fractions and eliminating duplicates, a total of 269 pitch classes are derived. The "issue" of unevenness, however, remains. The largest gap is now reduced to $23:24 \approx 73.7$c, found between $\frac{1}{1}$ and $\frac{24}{23}$ and their complements within the octave, and the smallest is $1681:1682 \approx 1$c, found between $\frac{41}{29}$ and $\frac{58}{41}$. Rather than arbitrarily filling in the larger gaps, as Partch decided to do, we proceeded by plotting the normalised SB-9 set (SB-9N) on the logarithmic spiral (Figure 11) to shed light on the problematic regions.

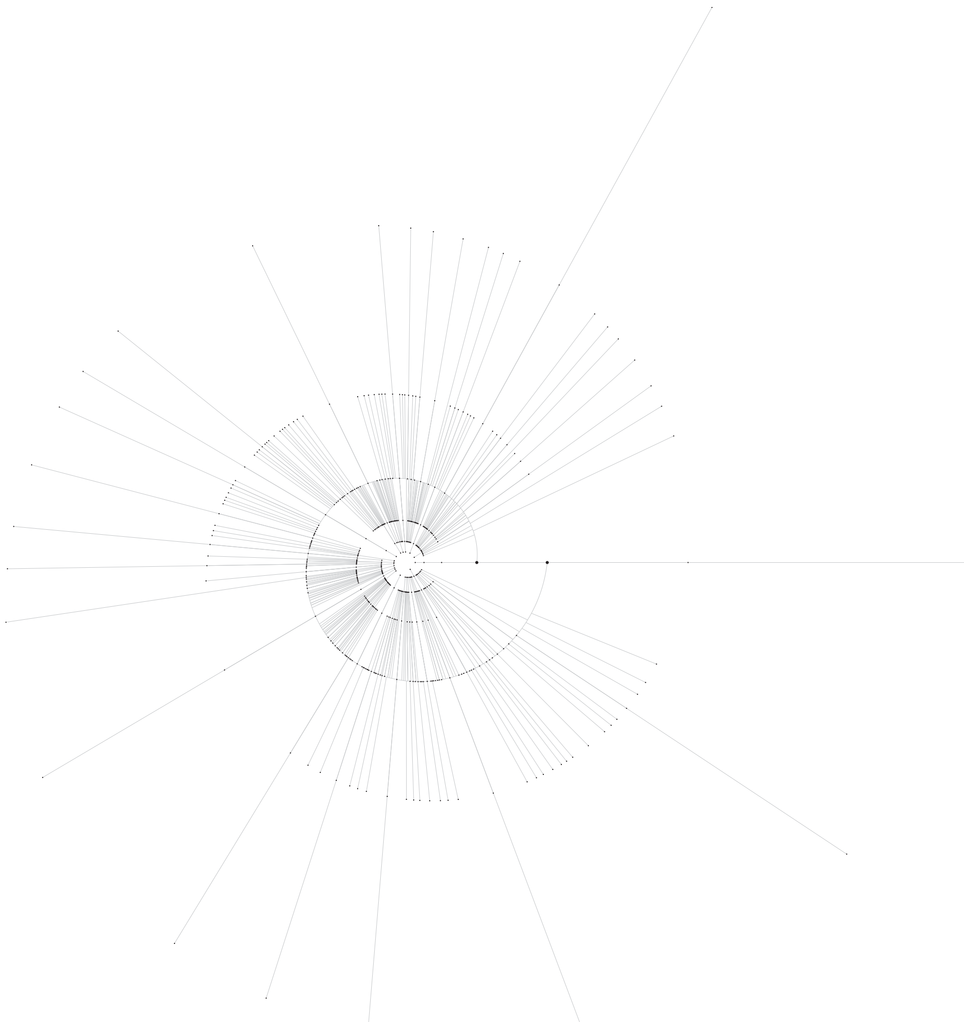


Figure 10: 511 fractions comprising the Stern-Brocot Tree through order 9, indicated as dots. Spokes are produced to the normalised forms, lying in the octave between $\frac{1}{1}$ and $\frac{2}{1}$, bounded by two larger dots indicated along the horizontal axis (= 0 radians).

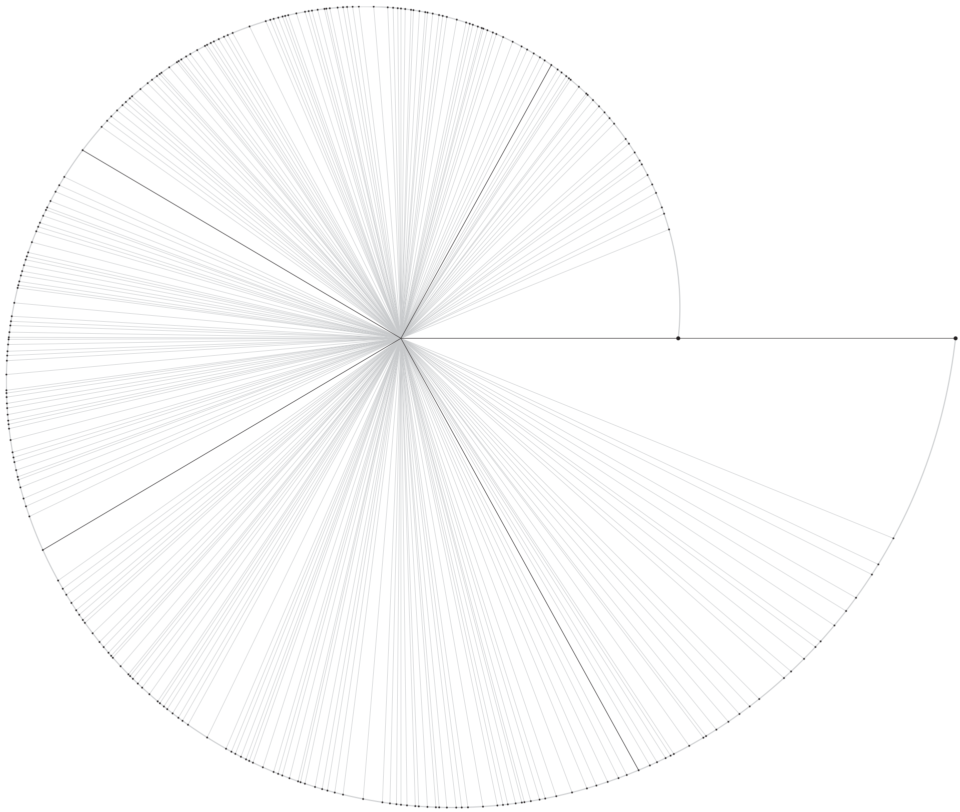


Figure 11: The 511 fractions of the Stern-Brocot Tree through order 9 render 269 pitch classes when normalised to the octave between $\frac{1}{1}$ and $\frac{2}{1}$ (abbreviated as SB-9N).

The largest gaps (in descending order) are found around the pitch classes $\frac{1}{1}$ (23 : 24 on either side), $\frac{3}{2}$ (104 : 105 ≈ 16.6c on either side), and $\frac{4}{3}$ (111 : 112 ≈ 15.5c on either side). Recall that among the conditions proposed for an enharmonically viable pitch set is an excellent representation of the extended Pythagorean and 5-limit, i.e. a chain of 53 perfect fifths with occasional “near-fifths” or “near-thirds”.[28] To achieve this, a pitch set needs to span several chains of 8 fifths, reaching schisma-altered thirds. In SB-9N the 3-exponents range between −3 and +3, spanning at most 7 perfect fifths. This suggests

[28] To keep the definition of tolerance in line with the previous discussion, in this case “near” refers to an alteration by about 2c or approximately one schisma. Note that the extended Pythagorean chain of 53 $\frac{3}{2}$'s produces the interval 284:353=19342813113834066795298816:19383245667680019896796723≈ 3.6c.

transposing the pitch set by additional perfect fifths, i.e. powers of 3, thereby extending the Pythagorean chain and reducing the larger gaps.

For other nearby Pythagorean pitches from the series of $\frac{3}{2}$'s, the gaps in SB-9N are as follows.

- $\frac{9}{8}$ (368 : 369 ≈ 4.7c)
- $\frac{16}{9}$ (351 : 352 ≈ 4.9c)
- $\frac{27}{16}$ (512 : 513 ≈ 3.4c)
- $\frac{32}{27}$ (1215 : 1216 ≈ 1.4c)

This final gap is once again smaller than one schisma.

To transpose the original normalised set of 269 pitches, the intervals $\frac{3}{1}$, $\frac{1}{3}$, $\frac{9}{1}$, and $\frac{1}{9}$ were chosen. These pitches, together with $\frac{1}{1}$, correspond to the notes of the Pythagorean pentatonic ♮C, ♮G, ♮D, ♮A, ♮E, which divide the octave into five parts relatively evenly[29] while at the same time corresponding to the five familiar open strings of the orchestral string instruments. The entire set of resulting fractions was normalised and duplicates were removed, resulting in what we call the "933-Tone-Tree Harmonic Space". By combining these pitches the gaps are evened out so the average step size is less than a schisma (i.e. ca. 1.3c). The complete, normalised pitch set typeset in HEJI notation is presented at the end of this article, and is available as a PDF for free download from plainsound.org.

This 933-Tone-Tree Harmonic Space, based on permutations of SB-9N, satisfies the remaining conditions for an enharmonic set of pitches (outlined above) in the following ways.

29 Building powers of 3 symmetrically around $\frac{1}{1}$, the first set of pitches, normalised, is $\frac{4}{3}$, $\frac{1}{1}$, $\frac{3}{2}$, dividing the octave into 2 perfect fourths (498c) and a tone (204c). The next set of ratios (pentatonic) divides the octave into 3 tones (204c) and 2 Pythagorean minor thirds (294c); these two intervals are relatively close in size. The next division (heptatonic) divides the octave into five tones (204c) and two limmas (90c), which are (once again) further apart. Further divisions introduce more intervallic variety (first the apotome, then the Pythagorean comma). Thus, the most "even" division among these options is the pentatonic.

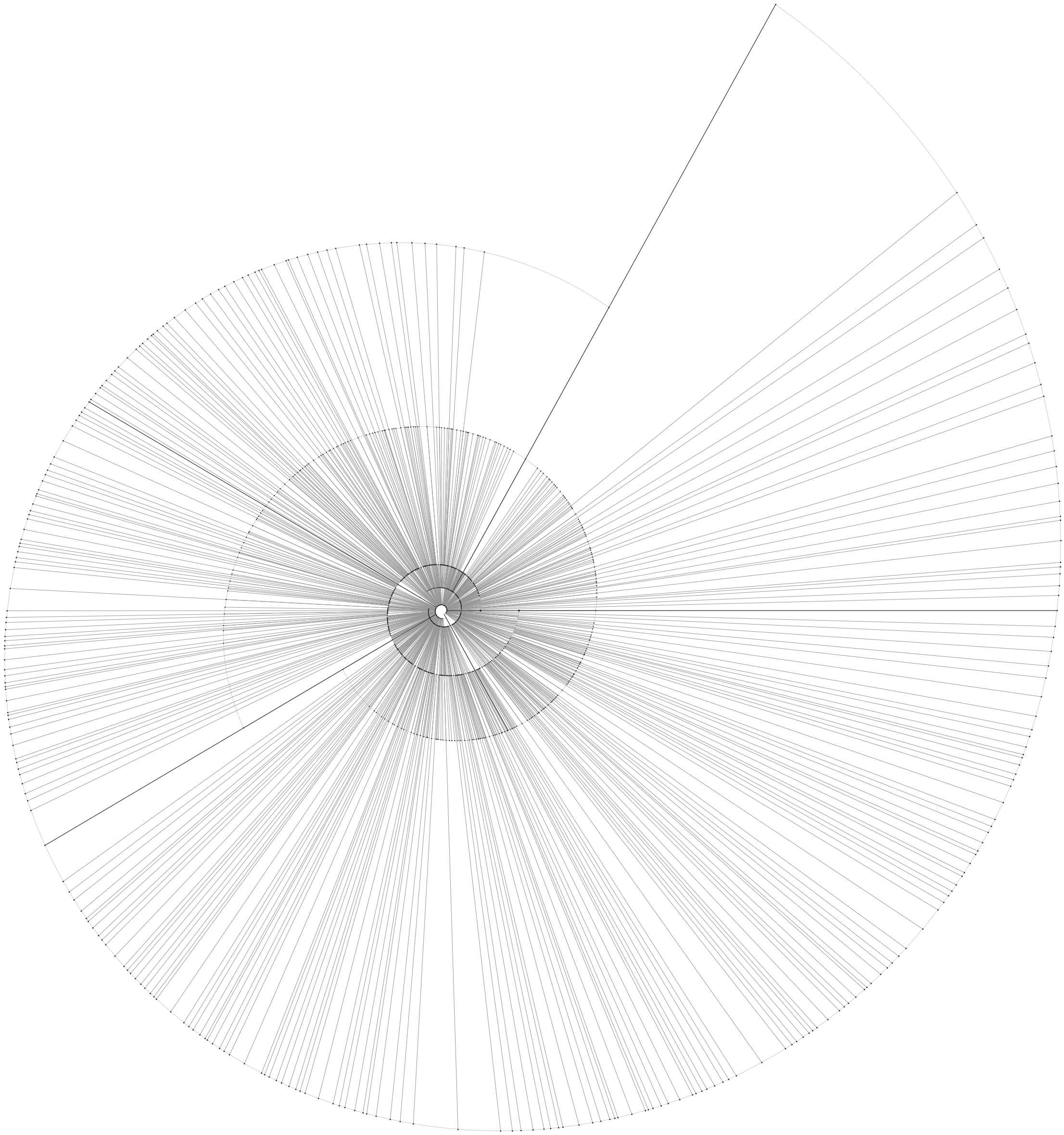


Figure 12: The 933-Tone-Tree Harmonic Space as a spiral, indicating the genesis of every pitch class. Each transposition of SB-9N is graphed as a set of points, forming one of five spiral segments, each delineating its own normalised octave. All of the pitches generated are produced as spokes leading to normalised pitches in the main octave $\frac{1}{1}$ to $\frac{2}{1}$. Comparing spokes and dotted points along the original, untransposed SB-9N segment between $\frac{1}{1}$ and $\frac{2}{1}$ demonstrates how the various transpositions combine to fill the "gaps" in the original set.

Pythagorean and 5-limit representation

The 43 pitches comprising the (3.5)-subgroup of SB-9N form a 41-tone Euler Lattice around ♮D with two schisma-differing pairs (♭E:♯D and ♭D:♯C). This set alone does not suffice as a "cycle" of fifths without a "wolf": the interval at the extremes is ♭B:♯F = 6561 : 10000 ≈ 729.6c, about a septimal comma larger than 2 : 3. To enharmonically extend the lattice, observe that the difference between the syntonic comma 80 : 81 (connecting 3 and 5) and the 41-comma 81 : 82 (connecting 3 and 41) is only 6561 : 6560 ≈ –0.3c.

By including the 24 additional notes completing the (3.5.41)-subgroup of SB-9N, the lattice may be extended to 51 notes, since 14 of the 24 are in fact enharmonically proximal to the 5-limit lattice already established. The enharmonic proximities include 6560 : 6561 ≈ 0.3c (e.g. ♮D:-D) or 1024 : 1025 ≈ 1.7c (e.g. -C:♯B). The 41-limit pitches form two 5-limit sublattices amongst themselves, retaining the structure of interlocking just major/minor triads that characterise the Euler Lattice.

Since the (3.5.41)-subgroup does not yet complete a cycle of 53 near-$\frac{3}{2}$'s, one more enharmonic proximity is needed. The best option involves including 10 notes of the 40-note (3.5.13)-subgroup of SB-9N. 13° differs from the Pythagorean by 26 : 27, which, divided harmonically in three, produces the harmonic series segment 78 : 79 : 80 : 81. Thus, three syntonic commas and the tridecimal thirdtone are close enharmonic proximities. The relevant enharmonic steps are either 1600 : 1599 ≈ –1.1c (e.g. -♮A:♮A) or at the "seam" connecting the extremes, 625 : 624 ≈ –2.6c (e.g. ♮C:♮B). In this case, the enharmonic "near-fifth" closing the circle is only very slightly narrower than the fifth in 12-EDO, resulting in a chain of fifths with no discernable "wolf". Since the full (3.5.13)-subgroup extends around this "seam", it is conceivable that an actual "mistuning" of the fifth may be unnecessary or avoidable in musical practice.

This builds a kind of adaptive just intonation framework similar to Oettingen's instrument mentioned above, and at the same time links to higher prime partials compactly related to the central reference note ♮D. Note that the "seam fifth" is actually closer to a 2 : 3 ratio than the "near fifth" produced when generating a chain of 53 pure fifths, and significantly better than the "near fifth" connecting the extremities of Oettingen's tuning. For a lattice diagram depicting this extension of the Euler Lattice, see Figure 13..

The Euler Lattice is based on an interlocking of notes in the relationship 4:5:6 (major triad) and its inversion 10 : 12 : 15 (minor triad). Extending this to include the most salient tuneable relationships (odd partials up to 19° and down to u19) produces the tonal map of enharmonic proximities to the extended Pythagorean shown in Figure 14 and Figure 15. The central axis here indicates pitches along the Pythagorean series, and the numbers along the left column indicate how many fifths away from $\frac{1}{1}$ these respectively lie, ranging from –28 to +28 fifths. Grey notes in square brackets fall outside the pitch set. Larger boxed regions indicate how the series of fifths nearly "folds" upon itself.

Tonalities

The various columns indicate deviations, in cents, from the extended Pythagorean, ranging from –4.2c to +4.2c. Notes above each other are in exact 2:3 ratio. If there is a dashed line *above* a note, its odd harmonic partials 3° and 5° are *exactly* represented within the 933-Tone-Tree Harmonic Space. If the line is solid, then additional odd partials up to 19°, but not all, are also exactly represented. If the letter "o" is indicated on the upper right side of the note, then the *entire* set of odd harmonics is present, and that note is an *otonality*. A similar principle holds with a dashed or solid line *below* a note, but in this case it refers to odd *subharmonic* partials down to u19. The letter "u" on the lower left side of the note indicates a *utonality*. An exclamation point beside a note indicates that a nearly exact enharmonic substitution *must* be made for partials 3 or 5 (when line is dashed) or for higher partials (when line is solid). In particular, this tonal map allows different substitutions to be considered based on the desired context and progression of pitches when remaining within the extended Pythagorean framework and optimises common-tone stability across enharmonic modulations: this is a feature unique to this pitch set, not found in adaptive JI.

In the complete table of 933 pitches, additional "tonalities", which are not in direct proximity to the 57 Pythagorean fifths mapped, are also included to complete a total of 98 possible overtonal/undertonal generators. Five of these, namely those based on the Pythagorean pentatonic, produce both "o-" and "u-" tonalities. Note the diagram on the second page (from 0 to +28 fifths) is a symmetric inversion of the previous page (from –28 to 0 fifths). Otonal structures are mirrored as utonal structures and vice versa.

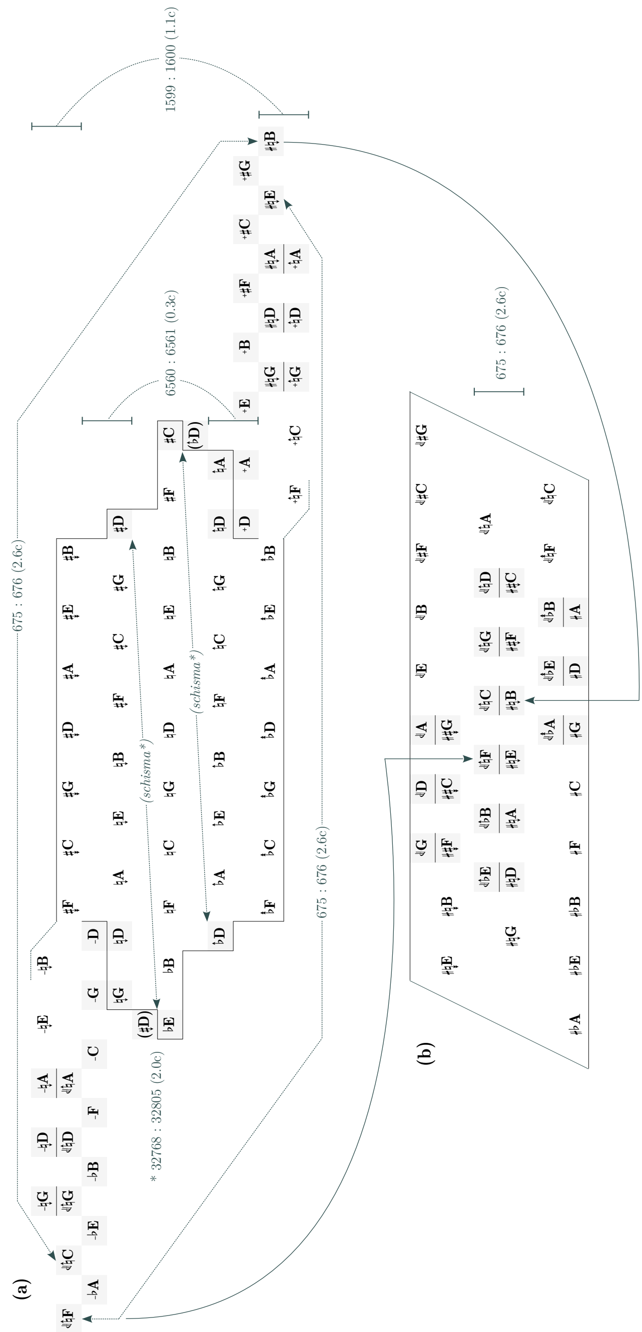


Figure 13: The enharmonically extended Euler Lattice in SB-9N.

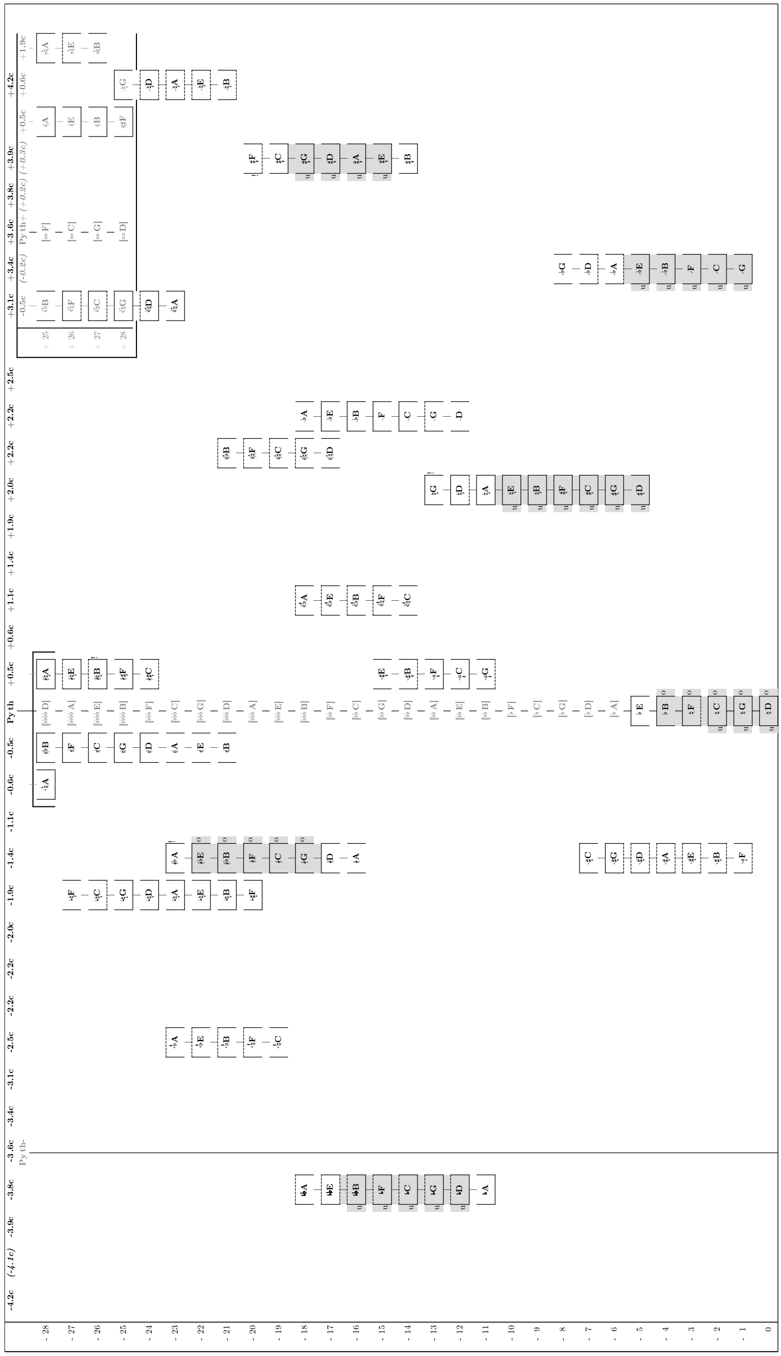


Figure 14: Enharmonic proximities to the extended Pythagorean (from −28 fifths to 0 fifths).

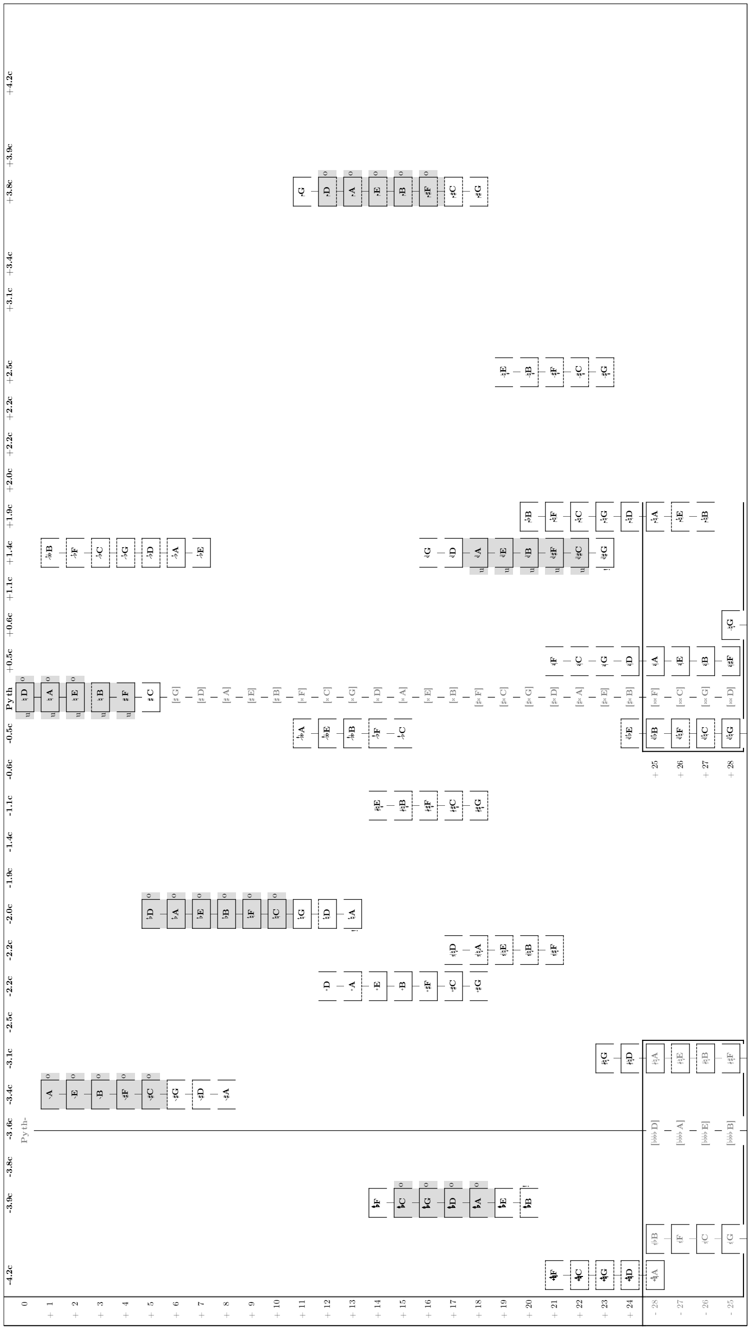


Figure 15: Enharmonic proximities to the extended Pythagorean (from 0 fifths to +28 fifths).

23-limit tuneable intervals

When the measure of adjusted pitch-class harmonic distance is applied on the pitch set (ignoring any differences caused by the reference pitch being ♮C, ♮G, ♮D, ♮A, or ♮E), the 226 pitch classes with least HD (those below a cut-off at 7.9) reproduce all of Sabat's 23-limit tuneable intervals except $\frac{23}{5}$, which is arguably the most difficult to perceive. These tuneable notes from the open strings of orchestral string instruments are indicated in the following chart of pitches as boxed ratios.

Systematic mapping of arbitrary input frequencies

Since the Stern-Brocot Tree is by construction a binary search tree, it is straightforward to use it to construct a search algorithm to find the "simplest" ratio arbitrarily close to any input frequency. This method may be adapted to the transposed, normalised pitch set derived in this paper. The frequency must simply be considered at *each* of the possible octave transpositions for which normalisation would provide a ratio (between $\frac{1}{9}$ and $\frac{9}{1}$), and with respect to each of the five Pythagorean transpositions of SB-9N. The ratio with the least harmonic distance within the desired search radius provides the result. Further analysis of the pitch set will require a comparative study of its "compactness" with respect to various measures, bounds, tolerances, and limits upon harmonic space. One of the significant areas of ongoing research is to explore how these properties might apply to the practice of composition with respect to microtonal just intonation.

Postlude: Some reflections on enharmonic relationships

In terms of progression and modulation within harmonic space – particularly within a complex microtonal framework – the importance of *common tones* as a practical aid for both performer and listener can hardly be overstated. Such tones serve as reliable points of reference between musicians *tuning in real time*. Furthermore, common tones help to guide a listener's ear through potentially unfamiliar webs of tonal relations, offering momentary flashes of familiarity. This interplay between the known and unknown may produce a surprising, even magical effect. As Jean-Philippe Rameau writes, the "moment of surprise passes like a flash, and soon this surprise turns into admiration, at seeing yourself thus transported, from one hemisphere to another, so to speak, without having had the time to think about it."[30]

To perfectly maintain common tones throughout an entire piece of music, however, has the potential to be problematic, as the number of distinct pitches can quickly compound beyond the abilities of conception and/or notation (i.e. so-called comma "pumps", which

[30] Deborah Hayes, "Rameau's Theory of Harmonic Generation: An Annotated Translation and Commentary of *Génération Harmonique* by Jean-Philippe Rameau" (PhD thesis, Stanford University, 1968), 179.

modulate a repeating passage upward or downward by a microtonal interval at each recurrence; see also the discussion of "free style" and "strict style" above). Even a simple four-chord progression might require an infinitude of distinct tones and signs if all common tones are perfectly maintained and the progression repeats endlessly, as Giambattista Benedetti famously brought to Cipriano de Rore's attention in a 17th-century letter (see Figure 16).

Figure 16: Two cyclic comma modulations transcribed in HEJI notation.

To manage this situation in such a way as to alleviate the need for what may seem to be infinite pitches while nevertheless maintaining justness of intonation, that is, by not introducing irrational *temperament* into the harmonic space, one might look toward *enharmonic* solutions. These are multiple harmonic branches or streams that very nearly coincide in absolute pitch height. The aforementioned schisma is just one clear historical example. Given a reference of ♮A (440 Hz), the notes ♯C (550 Hz) and ♭D (549.4 Hz) are nearly indistinguishable with acoustic instruments and in the majority of contexts, despite having completely different harmonic geneses: ♯C results from a single otonal step in the 5-dimension (with a ratio of $\frac{5}{4}$) while ♭D results from 8 utonal steps in the 3-dimension (with a ratio of $\frac{8192}{6561}$). This *enharmonic proximity* suggests a potential interchangeability in composition, where what seems *perceptually* to be a common tone may facilitate a progression or modulation along a new branch in harmonic space.

The 933-Tone-Tree Harmonic Space offers an ideal framework for taking advantage of enharmonic proximities in this manner, particularly as a basis for e.g. a computer-controlled keyboard instrument. This space presents a variety of rather novel enharmonic steps, most of which are smaller than the schisma. The distribution and occurrence of the most fundamental enharmonic connections in the pitch set are shown in Figure 17. The enharmonic connection 1215 : 1216 is by far the most prevalent, occurring 36 times between adjacent pitches. This suggests a deep, underlying connection between the 5- and 19-limits, respectively, at the heart of the pitch space (e.g. between ♯A and ♭B). The next most prevalent enharmonic is 1664 : 1665 (22 times) – a connection involving 5, 13, and 37 (e.g. between ♮C and ♮B).

In particular, such enharmonic proximities link many of the richest full and partial tonalities available in the set (as discussed above), and thus serve as an essential gateway to the extensive variety and tonal flexibility underlying the 933-Tone-Tree Harmonic Space. At this point, an obvious question is whether and exactly how this particular feature might be implemented in a systematic and contextually aware way. This would be the first step towards developing a kind of computer-controlled continuo instrument for real-time music making in just intonation, potentially for performing music in ensembles with traditional acoustic instruments: a new instrument that strikes a balance between flexibility, variety, manageability, and justness, a machine capable of learning how musicians choose to make logical enharmonic adjustments and choices at a very fine harmonic level.

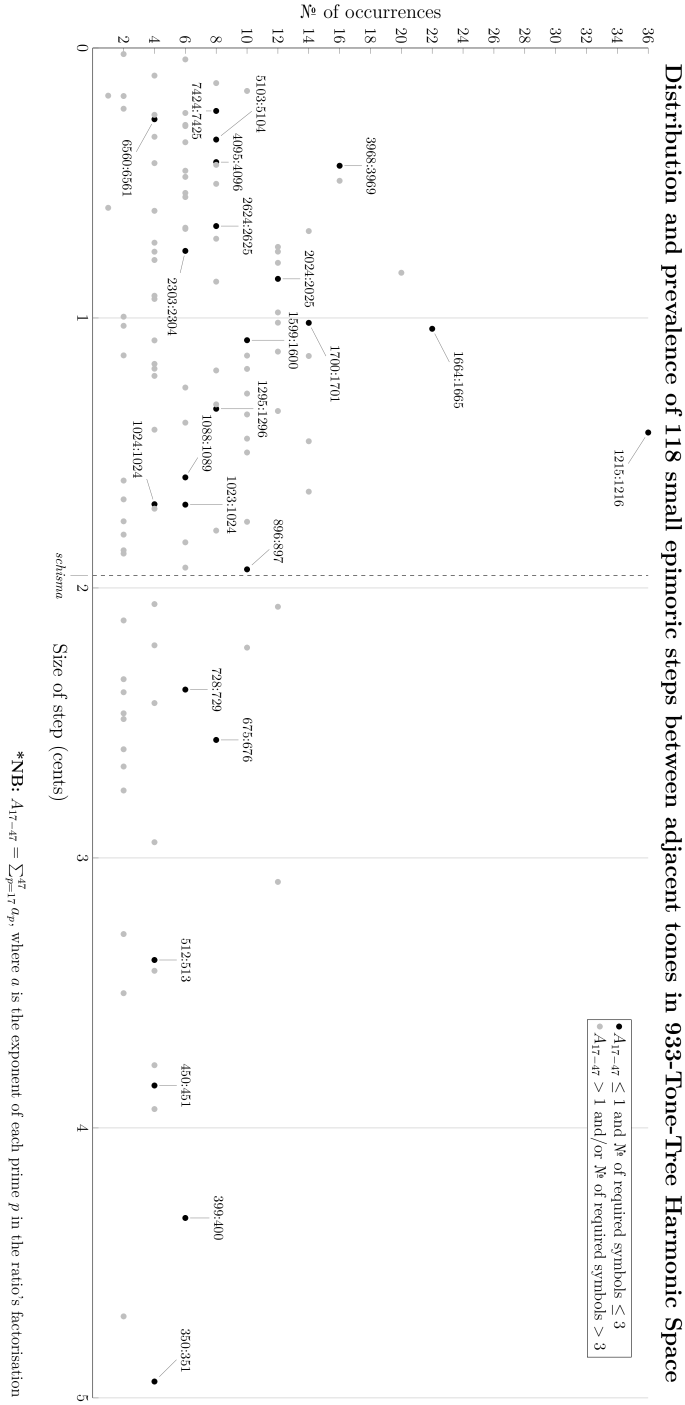


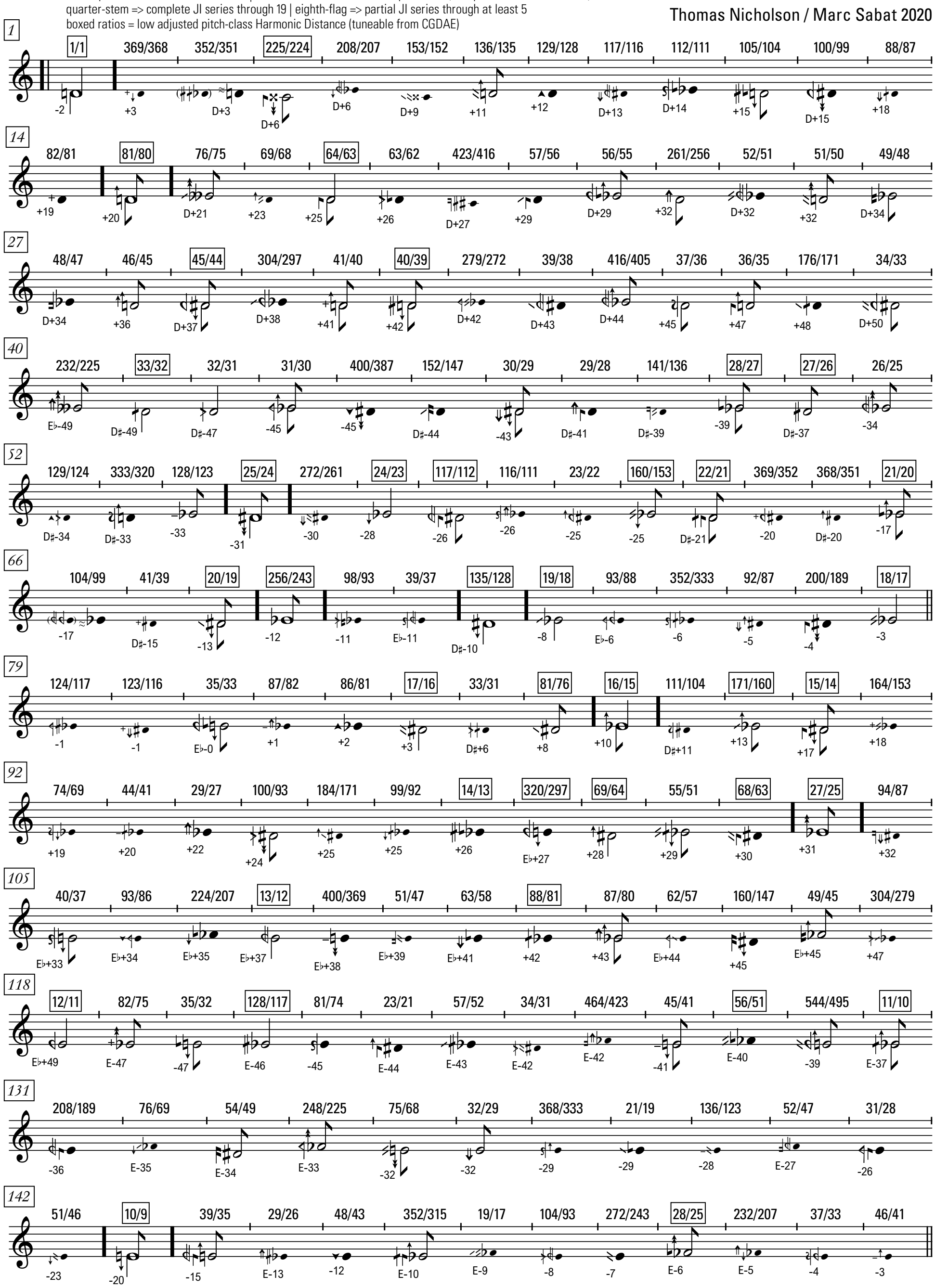
Figure 17: Enharmonic proximities in the 933-Tone-Tree Harmonic Space.

933-Tone-Tree Harmonic Space for James Tenney

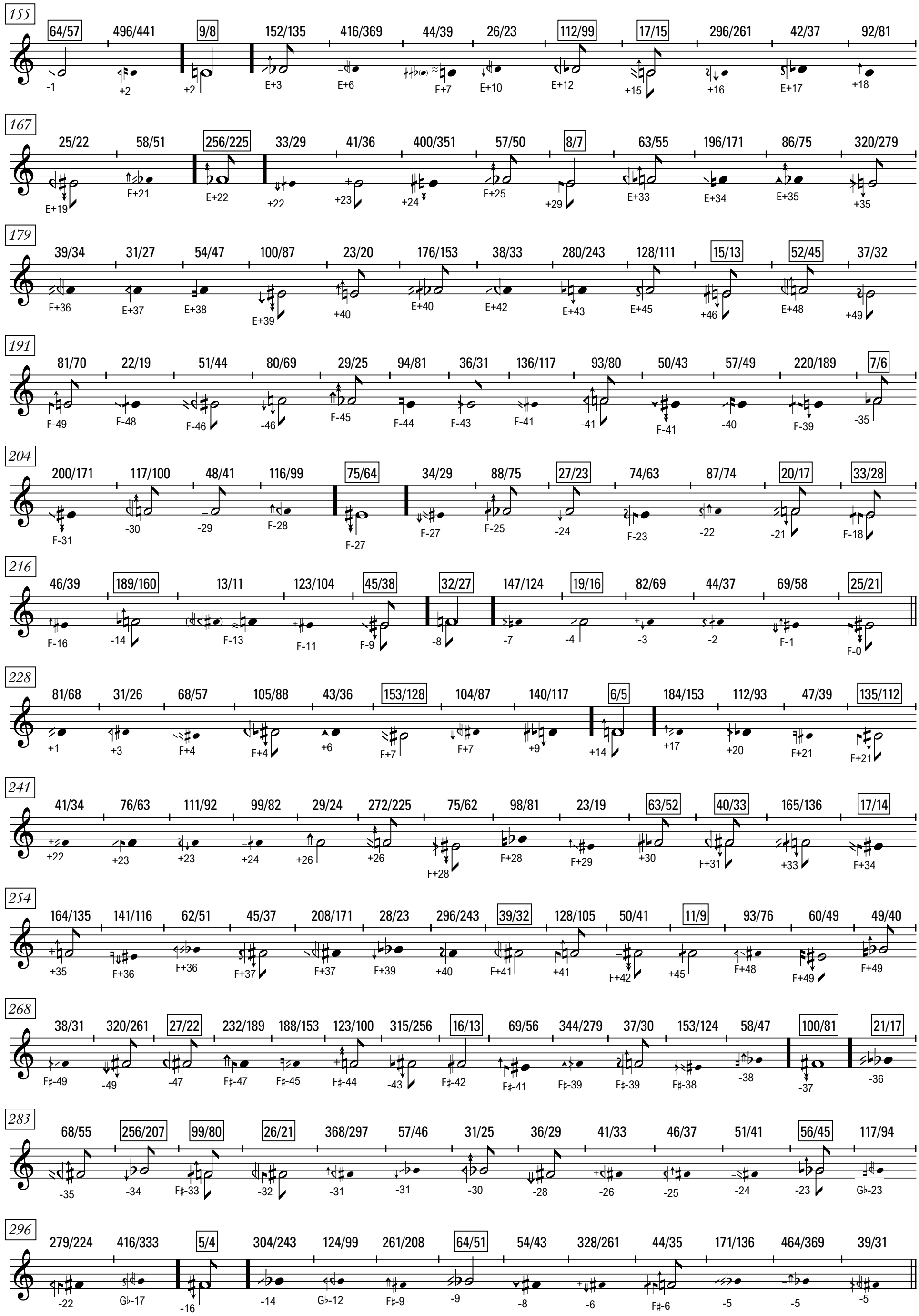
a compact and evenly distributed 47-limit JI pitch-set generated from a complete 9th order Stern-Brocot Tree
by combining and normalising the 511 ratios taken as intervals from each of the Pythagorean notes C-G-D-A-E

stem up = otonal, stem down = utonal | smaller noteheads combine 2 primes >7 (exception: 17 / 19 with 11 / 13)
quarter-stem => complete JI series through 19 | eighth-flag => partial JI series through at least 5
boxed ratios = low adjusted pitch-class Harmonic Distance (tuneable from CGDAE)

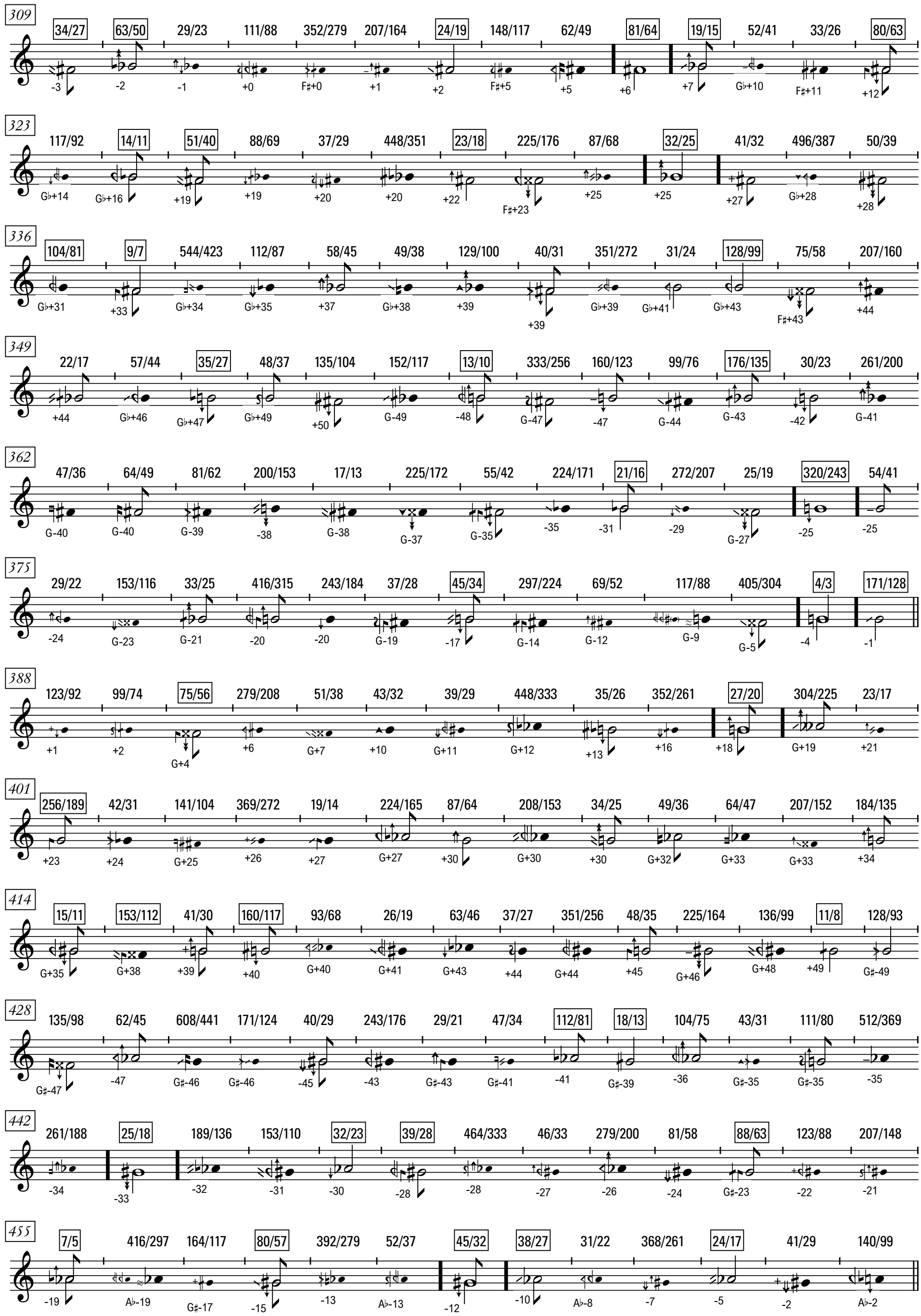
Thomas Nicholson / Marc Sabat 2020



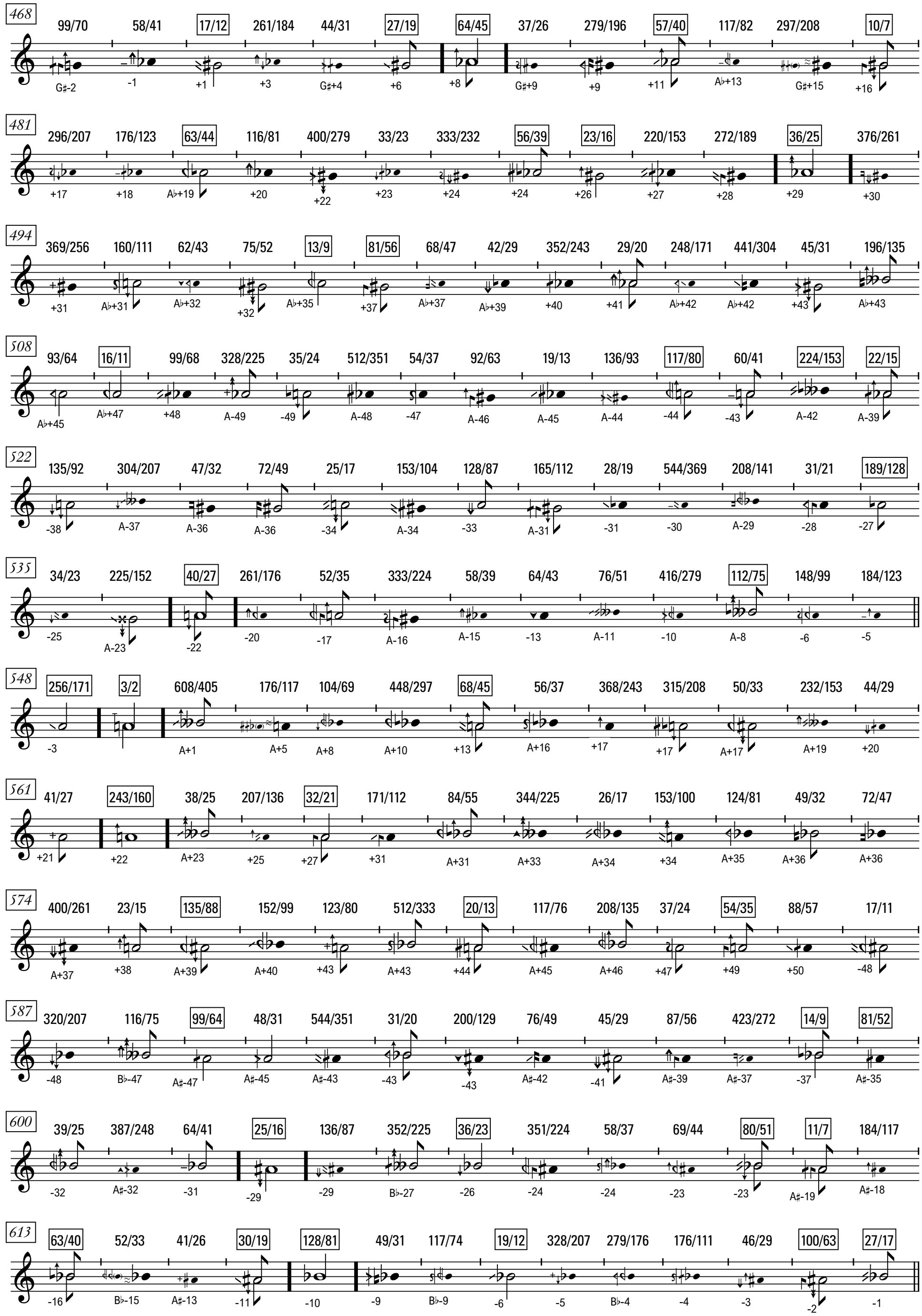
933-Tone-Tree Harmonic Space for James Tenney (Nicholson / Sabat 2020)



933-Tone-Tree Harmonic Space for James Tenney (Nicholson / Sabat 2020)



933-Tone-Tree Harmonic Space for James Tenney (Nicholson / Sabat 2020)



933-Tone-Tree Harmonic Space for James Tenney (Nicholson / Sabat 2020)

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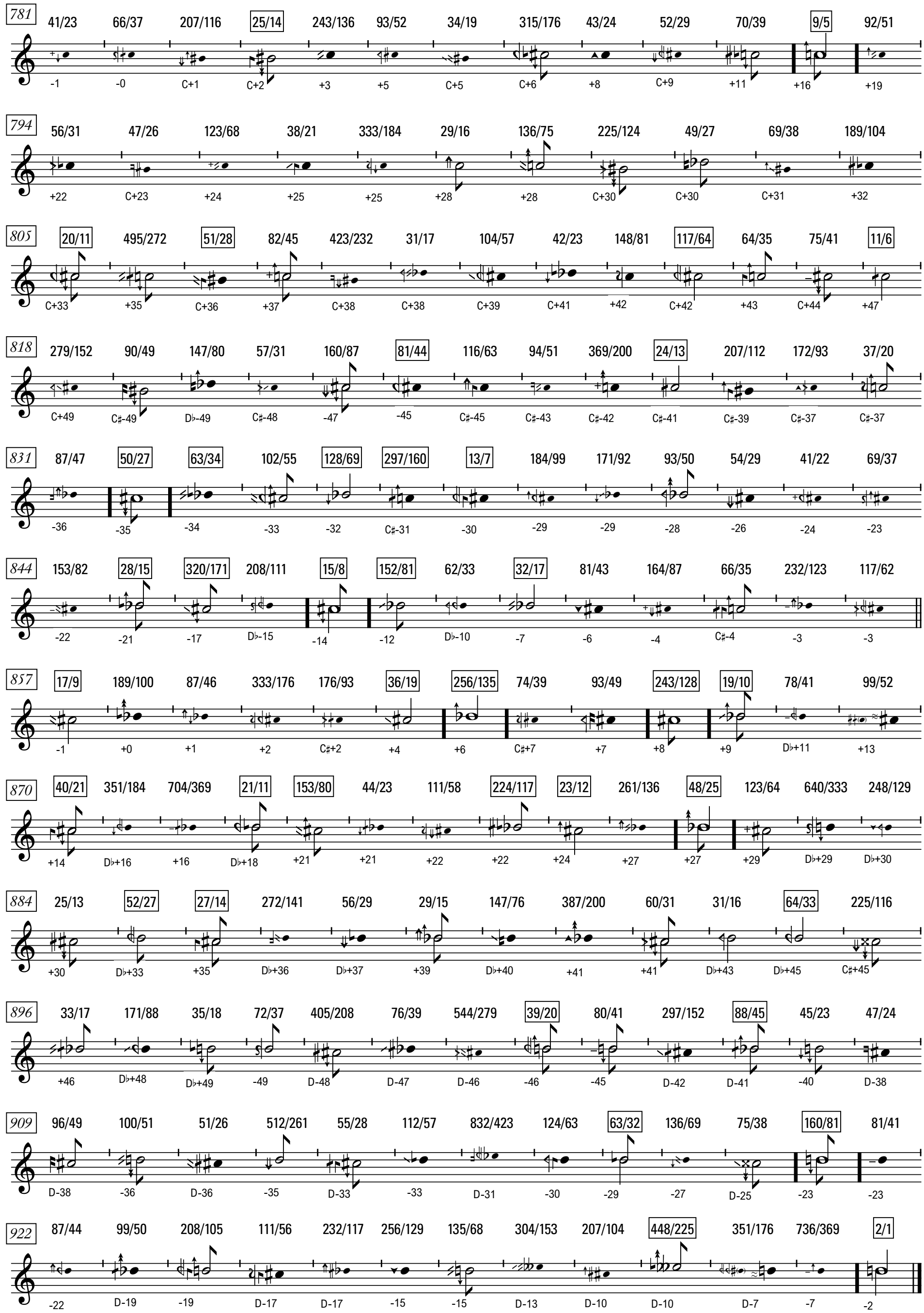

Bibliography

Barker, Andrew. *Greek Musical Writings, Volume II: Harmonic and Acoustic Theory*. Cambridge: Cambridge University Press, 1997.

Benedetti, Giovanni Battista. *Diversarum Speculationum Mathematicarum et Physicorum Liber*. Turin, 1585.

Erlich, Paul. “Harmonic Entropy”, 2004. http://www.soundofindia.com/showarticle.asp?in_article_id=1905806937.

Euler, Leonhard. *Tentamen Novae Theoriae Musicae Ex Certissimis Harmoniae Principiis Dilucide Expositae*. Ex typographia Academiae scientiarum, 1739.

Hayes, Deborah. “Rameau’s Theory of Harmonic Generation: An Annotated Translation and Commentary of *Génération Harmonique* by Jean-Philippe Rameau”. PhD thesis, Stanford University, 1968.

Helmholtz, Hermann von. *On the Sensations of Tone as a Physiological Basis for the Theory of Music*. Translated by Alexander J. Ellis. Third English edition. New York: Longmans, Green & Co., 1895.

Lang, Klaus. *Auf Wohlklangswellen durch der Töne Meer: Temperaturen und Stimmungen zwischen dem 11. und 19. Jahrhundert*. Graz: Institut für Elektronische Musik (IEM) an der Universität für Musik und darstellende Kunst in Graz, 1999.

Schweinitz, Wolfgang von. “The Classical Indian Just Intonation System”, 2007. https://www.plainsound.org/pdfs/srutis.pdf.

Tenney, James. *From Scratch: Writings in Music Theory*. Edited by Larry Polansky, Lauren Pratt, Robert Wannamaker, and Michael Winter. Champaign: University of Illinois Press, 2015.

Wilson, Erv. “The Harmonic Series as a Logarithmic Spiral”, 1965. http://www.anaphoria.com/harm-subharm.pdf.

Thomas Nicholson is a Canadian composer, keyboardist and violinist. His compositions examine the entrancing interactions between contrapuntal lines and harmonic fields (melodies and chords) under the influence of microtonal extended just intonation. In particular, he explores the musical consequences of isolating various minuscule, yet remarkably expressive enharmonic connections inherent to extended just intonation.

Marc Sabat is a Canadian composer of Ukrainian descent, based in Berlin since 1999. He makes concert and installation pieces, drawing inspiration from investigations of the sounding and perception of just intonation, and relating to various music forms - folk, experimental and classical. He is a frequent collaborator, seeking points of shared exploration and dialogue between various forms of experience and different cultural traditions.

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