Chords, Melodies: A Look at Harmony by Numbers; Part I: Using *Harmonic Radius* to Compare Rational Pitch Collections

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Marc Sabat

Abstract: This paper introduces the *harmonic radius*: a novel perspective on comparing the relative harmonicities of pitch collections tuned in rational intonation (JI). By means of a simple calculation that may be estimated in real time while playing, it enables musicians to further explore sounds of microtonal JI, finding connections between intonation and the psychoacoustics of harmony.

James Tenney's *harmonic space* maps rational frequency proportions in a lattice and defines a measurement called *harmonic distance*. The coordinates of a ratio are defined by its unique prime factorisation, which also establishes its *prime limit*. The exponents of a ratio's prime factors are interpreted as a vector of coordinates measured along axes representing the primes.

Given any finite collection of ratios in harmonic space, their common point of reference may be transposed so that all pitches have only *positive* exponents. The intervals between them remain the same, but now the pitches are expressed as natural numbers – as partials of their *nearest common fundamental*. This allows for a generalisation of harmonic distance applicable to any number of tones. By measuring an overtonal set of harmonic partials, harmonic radius maintains correlation with perceived harmonicity.

Using various forms of harmonic radius, it is possible to generate and compare intervals, chords, melodic gamuts and compact otonal subsets of harmonic space that include higher primes, while optimising possibilities for modulation and harmonicity.

Keywords: Harmonic radius; harmonicity; rational intonation; just intonation; microtonal composition; harmonic space; Tenney *harmonic distance*; Barlow *indigestibility*; Euler *Gradus Suavitas*

0. Introduction

This paper introduces a novel perspective on comparing the *relative harmonicities* of pitch collections tuned in rational intonation (JI).[1] By considering sets of partials related to a common fundamental, the idea of "pitch distance"[2] is generalised to *harmonic radius*[3] by a simple calculation consistently applicable across any number of pitches. Intervals, chords, scales, gamuts can be compared in an intuitive and musically fruitful way. Radius may be estimated in real time while playing, enabling musicians to further explore sounds of microtonal JI, finding connections between intonation and the psychoacoustics of harmony.

When taking on music that moves freely between many different frequencies and frequency relationships, composers, players, and listeners seek ways to navigate a space that can quickly become filled with a profusion of notes and interval sizes. The aim of this research is to move towards a quantitative and psychoacoustically based description of interval and chord qualities, building on James Tenney's concept of *harmonic space*,[4] with its associated *harmonic distance* measure, and an empirically determined set of *tuneable intervals*.[5]

The idea of *octave equivalence* is based on the property that any partials related by powers of 2 remain consonant with each other and with their fundamental. Octave transpositions do not create new microtonal collisions; instead, they transform chords into different voicings and inversions related by a common sonorous identity. Since the set of *odd partials* includes all harmonically related *pitch classes*,[6] its subsets present the most concordant[7]

1 The term *relative harmonicity* is used here to refer to fusion, i.e., the sensation that pitch relationships are aligned as partials of a common fundamental, enabling individual tones to meld into a single timbre. Rational or just intonation (JI) defines and compares pitches by using frequency ratios. In practice, it is usually acknowledged that ratios apply within a range of *tolerance*, so numerically large fractions may often be perceived as nearby "simpler" fractions. Note that the Xenharmonic Wiki article on "Chord complexity", https://en.xen.wiki/w/Chord_complexity, describes a mathematically related approach to comparing JI chords.

2 Measuring pitch in octaves, using units of $\log_2$(frequency): see Section 1 below.

3 See Equations 17–20 in Section 5 below.

4 James Tenney, Larry Polansky, Lauren Pratt, Robert Wannamaker, and Michael Winter, *From Scratch: Writings in Music Theory* (Champaign: University of Illinois Press, 2015).

5 Marc Sabat and Wolfgang von Schweinitz, "The Extended Helmholtz-Ellis JI Pitch Notation", 2005, masa.plainsound.org/pdfs/TIab.pdf. A selection of rational intonation dyads ranging from unison to triple octave, chosen by evaluating each interval between the first 28 harmonic partials to determine whether its tuning may be accurately established by listening for beating, periodicity and combination tones. Tests were conducted informally and empirically on string instruments, upward and downward from reference pitches in the alto range (220–440 Hz). See Table 2 for a comparison of various interval measures.

6 Every pitch class presented by a single harmonic series may be expressed as an odd-numbered partial, which also represents its first occurrence in the series, i.e., the position closest to the fundamental.

7 The term *concordance* applied to a collection of pitches refers to the smoothness and pleasantness of their tonal interaction; similarly, *discordance* refers to roughness and unpleasantness. *Consonance-dissonance*, on the other hand, can be thought of as including both aspects of contextually established "musical consonance" and psychoacoustic "sensory consonance" (Terhardt 1984)121-137 (1976. For an overview of consonance-dissonance concepts, see also Tenney 1988.

voicings[8] of families of sonically similar structures. Partials may be treated as octave-equivalent pitch-classes by dividing out all powers of 2 until an odd number is obtained to calculate the *odd radius*.

Powers of prime factors greater than 2 eventually generate partials that differ by small microtonal intervals or *commas* from numerically simpler connections to the fundamental, producing dissonant roughness or beating. For example, the sequence of powers of 3 – partials 3°, 9°, 27°, 81°, ... – eventually differs by a quintal (syntonic) comma from partial pitch-class 5° (= 80°). Thus, for each odd prime involved in generating a collection of partials, the range of its exponents affects overall harmonic saliency.

Focusing on specific primes and limiting exponent ranges creates *prime subgroups*,[9] one common way of categorising rationally related pitch sets. These highlight particular tonal relationships, guiding and informing the composition of melodies and chord sequences. Divisibility of composite partials into various primes, which affects the tonal coloration of pitch collections, may be considered by evaluating the *prime limit radius* of partials and their largest prime factors, resulting in an ordering similar to Euler's *Gradus Suavitas*[10] or Barlow's *indigestibility*.[11]

Including the fundamental and least common multiple in radius calculations is also useful, accenting the symmetry between *otonal* and *utonal* constructions.[12] Considering the chord of successive difference tones may also help in assessing degrees of spectral fusion. Using harmonic radius in various contexts, it is possible to generate and compare intervals, chords, melodic gamuts and compact otonal subsets of harmonic space that include higher primes, while optimising possibilities for modulation and harmonicity. Further discussion focusing on applications of harmonic radius in musical practice will follow in a subsequent paper.

8 Chords of odd partials are voiced closest to their nearest common fundamental, equal to their greatest common divisor (GCD).

9 The concept of prime subgroups constrains rational intonation pitch collections by deciding which prime numbers may serve as factors of a pitch ratio.

10 Leonhard Euler, "E33 – Tentamen Novae Theoriae Musicae", 1739, http://eulerarchive.maa.org/backup/E033.html; Charles Samuel Smith, "Leonhard Euler's 'Tentamen Novae Theoriae Musicae': A Translation and Commentary", PhD thesis, Indiana University, 1960, https://www.proquest.com/docview/301839227/citation/CD33AECE7F4C4C96PQ/1.

11 Klarenz Barlow, *Von der Musiquantenlehre*. 1. Auflage. Köln: Feedback-Studio-Verlag, 2008.

12 The terms *otonal* and *utonal* were devised by composer Harry Partch to describe harmonic and subharmonic structures, inspired by Oettingen's idea of structural dualism in major/minor tonality. Using the least common multiple produces the same value for otonal and utonal constructions, even though there are often differently perceived degrees of harmonicity.

1. Harmonic series; Benedetti's measure of concordance for rational intervals

An harmonic series, as understood in music, is, strictly speaking, a *sequence* of frequencies, which are all integer multiples of a fundamental frequency F. It may be generalised as the product of F with a set of *partials*[13] written in the form $P°$ and consisting of all positive integers in increasing order:

$$F \cdot \{1°, 2°, 3°, 4°, 5°, 6°, \ldots\}. \tag{1}$$

The fundamental F is *identical* in frequency to partial 1°; ratios of partial numbers to each other are exactly correlated with corresponding ratios of frequencies. The term "partial" in this case does not refer to a specific spectral component of a given sound but rather to a *one-dimensional pitch relationship* $1 : P$. The value

$$\log_2 P \tag{2}$$

expresses, by using the logarithmic base 2, the quantity *pitch distance* or *pitch-height*, measured in octaves: namely, how many octaves[14] separate a given partial $P°$ from its fundamental, 1°.

If two different pitches – simple sine waves or rich spectra – are sounded, it is possible to consider the interval between them, which is defined by the ratio of their fundamental frequencies. If this ratio is also a *rational number* expressible in lowest terms,[15] $\frac{P_1}{P_2}$, then the interval is the same as the relationship between two harmonic series partials

$$F \cdot \{P_1,\ P_2\}. \tag{3}$$

The magnitudes of the numbers P_i indicate how *far* from the common fundamental they lie.

The Italian mathematician Giovan Battista Benedetti (1530–1590)[16] observed, in a letter addressed to composer Cipriano de Rore, ca. 1563, that the concordance of a rational interval expressed as a fraction in lowest terms could be measured by considering the product of its numerator and denominator:

$$P_1 \cdot P_2 \tag{4}$$

13 The notation $P°$ refers to the number P as a harmonic partial (Sabat, 2020).

14 1:2 ratios and parts thereof. Since this paper is primarily concerned with the relative harmonicities of simultaneously sounding pitches, as established by near-exact periodicity, i.e., following the proportions of a harmonic series, the psychoacoustic phenomena related to "stretched octaves" in perception of melodic interval sizes is not relevant here.

15 A proportion in lowest terms consists of two or more numbers that do not share any common factor.

16 Giovanni Battista Benedetti, *Diversarum Speculationum Mathematicarum et Physicorum Liber* (Turin, 1585), 487–493, https://www.mpiwg-berlin.mpg.de/resources/publications/books/science-court-society-giovan-battista-benedettis-diversarum

This quantity, called Benedetti distance or Benedetti height, also happens to be the first *common partial* if both partials are sounded as tones with a rich harmonic spectrum. Mathematically, it is their least common multiple (LCM).

Transpositions of a pitch by one or more octaves are often given the same name, or *pitch class*, based on a perceptual quality introduced above as *octave equivalence*. Within an harmonic series, octave transpositions are made by multiplying or dividing a partial by 2. The *lowest* occurrence of any pitch class within the series is therefore always an *odd*-numbered partial, which is not divisible by 2. Thus, to enumerate all the unique pitch classes presented by the series, it is sufficient to take the set of all odd-numbered partials.

2. Euler's *Gradus Suavitas*; Barlow's *indigestibility*

The observation that ratios of small numbers produce concordant intervals has been noted since ancient times. Such intervals, often tuneable by ear, have been combined to make various scales and modes (e.g., tertial or Pythagorean/Mesopotamian tuning; Greek, Arabic, Persian divisions of the tetrachord; Zarlino's just intonation scale; Bharata's vina experiment described in the *Natyasastra*).

The building blocks of ratios are numbers and their unique prime factors. Products of smaller prime numbers produce composite numbers that themselves remain relatively small. Many common harmonic tunings used in music are generated exclusively from the first three primes: 2, 3, and 5. In 1739, mathematician Leonhard Euler devised a method of depicting networks of such "5-limit"[17] or *quintal* relationships on a two-dimensional graph, providing a visualisation of pitch classes interlocking the partial 3° relationship (perfect fifths) and the partial 5° relationship (major thirds) in a lattice of triads (*Tonnetz*).[18]

In the same book, *Tentamen Novae Theoriae Musicae*, Euler also developed a method for evaluating the "pleasantness" of rationally tuned pitches, intervals and chords, which he called *Gradus Suavitas*. He assigned a degree to each natural number.[19] The degree of the number m represents the degree of pleasantness of the frequency ratio $1 : m$. The unison ratio $1 : 1$ is assigned degree 1 and the interval $1 : p$, where p is a prime number, is assigned degree p. The composite interval $1 : pq$, where p and q are both prime, is assigned degree $p + q - 1 = 1 + (p - 1) + (q - 1)$.

By induction, the following formula calculates the Euler degree of any number, based on its (unique) prime factorisation, i.e., when written as a product of powers of various primes –

17 The term *prime limit* defines the largest prime number that may be a factor of any element in a set of ratios.

18 Marc Sabat and Thomas Nicholson, "A Compact Enharmonically Viable Subset of Harmonic Space: The Stern-Brocot Tree and Some Thoughts About Lattices and Spirals", *Živá hudba*, 2021, https://ziva-hudba.info/stern-brocot-ji/.

19 Natural numbers are the positive integers or positive whole numbers.

$$GS\left(\prod_i {p_i}^{k_i}\right) = 1 + \sum_i k_i(p_i - 1) \tag{5}$$

– counting the number of each prime in the factorisation (k_i) and scaling by the prime's magnitude (p_i – 1). (The "–1" ensures that each power of the prime number 2 is counted once, thus keeping track of "octaves".) Furthermore, Euler defined a way of measuring the *Gradus* of any fraction and, by extension, of any rational chord written as a lowest terms proportion, by equating it with the LCM of the numbers:

$$\mathrm{GS}(a{:}\,b{:}\,c) = \mathrm{GS}\big(1{:}\,\mathrm{LCM}(a, b, c)\big) \tag{6}$$

As noted above in the discussion of Benedetti distance, this is also the *lowest common partial* of the pitches: it determines the frequency at which beating might occur if the chord is sounded and tuned with rich timbres.

By inventing a calculation providing a quantitative, graduated continuum of relative concordance, Euler's method supports the idea that intervals with prime factors greater than 5 might also possess a certain pleasantness. The musical role of proportions beyond the 5-limit had been documented by Greek and Arabic theorists (Claudius Ptolemy, Abu Nasr Farabi, Ibn Sina, among others), reflecting practices of their time and place,[20] but such sounds had only very rarely been used in European music.[21]

In Europe, the most prevalent temperament from the 16th to 19th centuries was 1/4-syntonic-comma meantone. It represents the 4:5 major third interval exactly as well as having two nearly exact 4:7's, written as "augmented sixths" (b♭–g♯, e♭–c♯). Some composers, among them Michelangelo Rossi and Girolamo Frescobaldi, made occasional use of these *septimal* sounds as musical consonances, composing chords combining 4:5:7 or 6:7:8.[22] Giuseppe Tartini argued theoretically for a broader inclusion of septimal consonances in figured bass progressions, inventing a microtonal notation and composing several examples (Johnson, 1985). Nevertheless, the musicality of higher primes continues to be debated. Many *tuneable* sounds including 7° exist, but the consonance

20 Andrew Barker, ed. *Greek Musical Writings. Vol. 2: Harmonic and Acoustic Theory*, Cambridge Readings in the Literature of Music (Cambridge: Cambridge University Press, 1989); Idin Samimi Mofakham, "Holographic Composition Technique", Norwegian Academy of Music no. 8 (May 2023). https://www.researchcatalogue.net/view/1092359/1871366.

21 Patrizio Barbieri, *Enharmonic Instruments and Music 1470–1900: Revised and Translated Studies*, Tastata 2 (Latina: Il Levante Libreria, 2008). A notable exception is the enharmonic music and instruments of Giovanni Battista Doni (1595–1647).

22 Thomas Ciszak, "Frescobaldi and the Natural Seventh", presented at Winter Musik, Akademie der Künste Berlin, 2022.

produced by their periodic signatures often combines strong timbral fusion and roughness in a distinctive way not always perceived as pleasant.

In 1978, composer Clarence Barlow modified Euler's formula. His equation for *indigestibility* (ξ) of a number slightly altered Euler's measurement scale by dropping the term "1+":

$$\xi\left(\prod_i {p_i}^{k_i}\right) = 2\sum_i k_i \frac{(p_i - 1)^2}{p_i} \tag{7}$$

Here, $\xi(1)$ takes the value 0 and $\xi(2)$ takes the value 1, counting the "number of octaves" more consistently, as powers of 2. In addition, Barlow multiplied Euler's scaling factor, $(p_i - 1)$, by an additional factor, $2\frac{(p_i-1)}{p_i}$. If p_i is 2, this factor is equal to 1. As p_i increases, this factor also increases, gradually approaching 2. Thus, it favours *smaller* primes. Barlow also derived a measure of an interval's *polarity.* For a ratio $p : q$, polarity is defined as

$$sign\big(\xi(q) - \xi(p)\big) \tag{8}$$

and the ratio's *harmonicity* is defined as the polarity value, –1, 0, or 1, multiplied by

$$\frac{1}{\xi(\mathrm{p}) + \xi(\mathrm{q})} \tag{9}$$

– for pitch sets (primarily scales and gamuts), *specific harmonicity* is evaluated by averaging harmonicities of all the pairwise intervals.

Note that the denominator expression $\xi(p) + \xi(q)$ is equivalent to $\xi(\mathrm{p}q)$. Like Euler, Barlow's expression equates the degree of a ratio in lowest terms with its LCM.

Benedetti distance evaluates the "size" of fractions based on the absolute magnitudes of numerator and denominator, without considering the size of prime factors. By contrast, Euler's and Barlow's measures consider the *degree of divisibility*: they order the natural numbers so that products of smaller primes (more easily constructed pitches) precede larger primes, mirroring how many musical tone systems are built up from a smaller set of generating concordances.[23]

23 See Appendix 1 for a table comparing the ordering of integers produced respectively by the Euler, Barlow, and Sabat measures.

3. Tenney's *harmonic space* and *harmonic distance*

In *John Cage and the Theory of Harmony* (1983) American/Canadian composer James Tenney formulated the concept of *harmonic space*, generalising Euler's lattice model. Tenney's harmonic space maps fractions, representing rational frequency proportions, in an n-dimensional lattice.

A ratio is assigned coordinates in the lattice. These coordinates are defined by the ratio's unique prime factorisation, which also establishes its prime limit.[24] The exponents[25] of a ratio's prime factors are interpreted as a vector of coordinates measured along axes representing the primes. The numerator comprises primes with positive exponents; the denominator, negative exponents; any excluded primes have exponent 0. Together, these integer exponents produce a vector of coordinates. The range along each prime axis is determined by the respective exponent's magnitude. The 2-axis is enumerated in octaves (1:2), the 3-axis in perfect twelfths (1:3), and so on. The scaling of each axis is in units of length $\log_2 p$, so that the usual measure of pitch-height in terms of octaves is preserved.

As an example, consider the diatonic semitone interval found between partials 15° and 16°, written as the ratio 15:16 or, equivalently, as the lowest terms fraction 16/15, representing the pitch one diatonic semitone above 1/1. If 1/1 is assigned a specific frequency, 16/15 determines a new frequency, i.e., a specific *pitch*:

$$\frac{16}{15} = 2^4 \cdot 3^{-1} \cdot 5^{-1} \tag{10}$$

In harmonic space this pitch is represented by the coordinates (4, –1, –1), describing a journey by intervals: taking four octaves $(\frac{2}{1})^4$ *upward* and one twelfth $(\frac{3}{1})^{-1}$ plus one seventeenth $(\frac{5}{1})^{-1}$ *downward* from the starting point $(\frac{1}{1})$. Upward steps, i.e., those with a positive exponent, multiply frequencies by powers of a prime number; downward steps, those with negative exponents, divide by the same prime. These steps can be taken in any order, changing the pitches encountered along the way. The combined pitch of two successive ratios is determined by adding their exponents; the interval between two ratios is determined by subtracting them. Thus, transposing the starting point (origin) is distance-preserving (isomorphic).[26]

24 Since the set of all prime numbers is infinite, usually the term "harmonic space" is taken to refer to a *subset* of the entire possible space. Dimensionality of the vectors making up a subset is then determined by a specific *finite* subset of primes. For example, one can define 47-limit-space, with coordinates for all primes up to 47 or (2, 3, 7)-space, which is three-dimensional.

25 The number of times each prime occurs in the product is called its power or exponent.

26 Each pitch's exponents are shifted by the same vector, so the difference between pitches remains unaffected.

Based on this idea of "moving" through harmonic space, Tenney defined a "city-block" metric called *harmonic distance*, which is the sum of the magnitudes of all the individual component steps needed to get from the origin (1/1) to the desired ratio (16/15) by a direct route. It does not matter whether these steps are taken upward or downward. Mathematically, this means that the *sign* of the exponents is ignored; only their magnitude or *absolute value* is considered.

In the Euler and Barlow equations shown above, the exponents n_i are always positive numbers, since GS and indigestibility are calculated for positive integers and then applied to fractions. In the harmonic distance equation that follows, the product on the left side can also be a *fraction,* so the expression potentially includes positive and negative exponents:

$$HD\left(\prod_i {p_i}^{k_i}\right) = \log_2\left(\prod_i {p_i}^{|k_i|}\right) = \sum_i \log_2({p_i}^{|k_i|}) = \sum_i |k_i|\log_2(p_i) \qquad (11)$$

Applied to 16/15 (skipping the third step in Eq. 11 for brevity), this equation gives a harmonic distance of:

$$HD(2^4 \cdot 3^{-1} \cdot 5^{-1}) = \log_2(2^4 \cdot 3^1 \cdot 5^1) = 4 \cdot \log_2(2) + 1 \cdot \log_2(3) + 1 \cdot \log_2(5) \approx 7.91$$

By taking the absolute value of exponents, harmonic distance in effect "flips" the denominator part of the initial fraction to the top:

$$HD\left(\frac{num}{den}\right) = \log_2\left(\frac{num \cdot den}{1}\right) = \log_2 num + \log_2 den \qquad (12)$$

The base 2 logarithm of the product of two partials "num" and "den" measures pitch distance in octaves and can be rewritten as a sum of the individual pitch distances of "num" and "den".

Taking once again the example 16/15, this alternate formulation gives:

$$HD\left(\frac{16}{15}\right) = \log_2(16 \cdot 15) = \log_2(16) + \log_2(15) \approx 7.91$$

Notice that, along the way, Benedetti's 16th-century measure of concordance makes a reappearance! Euler is also revisited: since the numerator and the denominator are assumed to be in lowest terms, their LCM (least common multiple) is equal to their *product*; Euler equated a fraction's pleasantness with its LCM:

$$GS\left(\frac{num}{den}\right) = GS\big(LCM(num, den)\big) = GS(num \cdot den). \qquad (13)$$

This gives another way of thinking about what harmonic distance is evaluating: it measures *one-dimensional pitch distance* from the fundamental to the lowest common partial of two pitches, represented by the numbers *num* and *den.*

4. Partch's *otonality/utonality*; Tenney's *intersection*; Erlich's *harmonic entropy*

Intervals, ratios of two frequencies, are inherently symmetric. If the order of frequencies is reversed, the magnitude of the pitch distance between them remains the same; only its "direction" changes. If frequency increases, the pitch distance is rising, or positive; if frequency decreases, the pitch distance is falling, or negative.

Chords, on the other hand, are made up of several intervals. Completely different sound constellations may be composed by reordering these intervals. Any sequence of intervals has two related but usually *sonically different* forms: an *upward* sequence, with all intervals taken as positive pitch distances, and its inversion, the same sequence projected *downward* with all intervals taken as negative pitch distances.[27]

Harry Partch proposed the terms *otonal* and *utonal* for these two related chord-forms.[28] In his model of tonal relations, Partch adapted concepts from theories of harmonic dualism, suggested by Jean-Philippe Rameau and developed by Moritz Hauptmann, Arthur von Oettingen and Hugo Riemann, among others. Originally conceived as a way of explaining the practice of triadic major and minor chord harmony, Partch extended these ideas to 11-limit hexads and devised a symmetrically constructed 43-tone rational pitch gamut.

Any pitch collection in his tone system may be mirrored by inverting the intervals.[29] Partch postulated that such a close *structural* relationship, which he demonstrated using a lattice diagram he called the "tonality diamond", would also be perceived as a *sonic* relationship: the harmonic series, with its major triad 4:5:6, reflected in a *subharmonic* series, containing the minor triad "1/4:1/5:1/6".[30] Partch's student Ben Johnston applied Partch's harmonic system in numerous pieces, often demonstrating unexpected connections between harmonic dualism and serial techniques based on tone row transformations.[31]

27 The upward and downward forms are identical if and only if the sequence of successive intervals forms a palindrome.

28 Harry Partch, *Genesis of a Music: An Account of a Creative Work, Its Roots, And Its Fulfillments*, 2nd enlarged edition (New York: Da Capo, 1974).

29 If the first pitch is the ratio b/a, then the mirrored form begins with 2a/b (reduced to lowest terms). By symmetry around 1/1, all subsequent intervals can also be found within the gamut by multiplying their reciprocal by 2, reducing, and normalising.

30 The fractions represent intervals *below* a common generating pitch, i.e., a common partial.

31 For example, in the *String Quartet No. 6*, built from interlocking harmonic and subharmonic scales.

However, this constructed symmetry is not paralleled in sound. The phenomena which establish harmonicity – combination tones, beating, periodicity, fusion – all depend on the correlation of frequencies that are *separated by equal differences*. Subharmonics invert the arithmetic division of frequency ratios, producing a proliferation of many *differing* differences. This often results in an harmonically distant and extremely low fundamental frequency; if written as an overtonal proportion, a utonal chord often includes larger numbers than its otonal counterpart.

The major triad 4:5:6, subharmonically inverted, becomes the minor triad with overtonal proportion 10:12:15. Both chords share the same common partial, 60, as well as the same fundamental (1) and the same three constituent intervals: 2:3, 4:5, and 5:6. But the difference tones between successive notes of the first chord are both 1, while the second chord differences are 2 and 3, and the two sounds have different degrees of harmonicity.

The two forms of the 11-limit hexad used by Partch as a basic chord differ even more. Its otonal form is 4:5:6:7:9:11, with least common multiple 13860. The utonal version of the same chord is 13860 divided by each of 11, 9, 7, 6, 5, and 4, giving the proportion 1260:1540:1980:2310:2772:3465. Even with a fundamental of 1 Hz, more than four octaves below the range of human hearing, this chord would be voiced well above the treble staff; it *cannot* be harmonically salient as a simultaneous sound. However, taken as a gamut, each of the intervals is a simple tuneable relationship. Together, these pitches produce a combinatoric kaleidoscope of varying harmonicities that may unfold and be experienced in time.

Nevertheless, by adopting the method of associating chords with their least common multiple, or by using averaging across all of the interval pairs, the methods suggested by Euler, Barlow, Tenney/Benedetti and Rafael Cubarsi[32] for evaluating collections of three or more pitches do not differentiate the relative harmonicities of otonal and utonal versions of the same chord.

In his 1979 text *The Structure of Harmonic Series Aggregates*, Tenney proposed a different measure he called *intersection*,[33] which compares the composite spectrum of a rational sound combination, extending up to its lowest common partial, with a complete harmonic series from the combination's nearest common fundamental.[34] This method assumes that each of the component sounds are timbres with full spectra comprising all harmonic partials (sawtooth waveforms or similar sounds). However, any chord that

32 Rafael Cubarsi, "Harmonic Distance in Intervals and Chords", *Journal of Mathematics and Music* (Society for Mathematics and Computation in Music) 13(1) (2019), 85–106, https://doi.org/10.1080/17459737.2019.1608600.

33 Tenney et al, *From Scratch*.

34 Equation in the case of two partials a and b: $\frac{a+b-1}{ab}$. When more pitches are involved, the equation becomes much more complex.

includes the fundamental, and therefore any interval of the form $1 : p$, obtains an intersection value of 1. If a composite sound includes all partials, its intersection is equivalent to that of a single, fused harmonic sound. Thus, among such chords, there is no quantitative correlate to differing perceived degrees of concordance. In addition, as Tenney himself states, calculating intersection in larger sets of pitches is algebraically unwieldy and requires time-consuming combinatoric computations.

More recently, in the 1990s, theorist and musician Paul Erlich, with the collective input of colleagues on the Mills College and Yahoo Tuning Lists, including (among many others) Steve Martin and Mike Battaglia, invented and developed a concept called *harmonic entropy*.[35] If pitches deviate slightly from rational proportions, the intervals and chords they produce in combinations with each other may be interpreted as slightly mistuned variations of various proximal rational structures. The detuning may be perceived as timbral coloration, modulation or "noise". Harmonic entropy seeks to model this psychoacoustic *tolerance* effect, deducing which structures are most likely to be perceived.

To do so, a statistical calculation is computed with respect to a set of weighted rational inputs, whose harmonicities and/or tolerances are estimated beforehand. In the case of intervals, Tenney/Benedetti (or Farey/Weil) distance is used to establish instantaneous scaling values ("harmonicities"), while the method of Farey Sequences,[36] i.e., computing mediants, is used to set boundaries between rational target intervals and determine their respective "ranges of harmonic influence" (or "tolerances").[37] In the case of arbitrary chords and aggregates, weightings and spheres of influence of selected rational chords must be approximated using other techniques, including, potentially, some of the methods proposed below.[38]

5. Harmonic Radius

This paper introduces a new quantity, *harmonic radius*, which is a modification and generalisation of Tenney's harmonic distance based on Benedetti. HD of a fraction in lowest terms is, in effect, a *sum* of two pitch distances, derived from two numbers, the numerator and the denominator, taken as partials of their closest common fundamental. In other words, it is *twice* the *average distance* to these two partials.

Harmonic radius expresses "how far", on average, a set of any number of harmonic partials lies from its common fundamental. Since harmonicity and fusion within the musically salient range of frequencies are modelled by the harmonic series and increase with

[35] William A. Sethares, *Tuning, Timbre, Spectrum, Scale*, 2nd ed. (London: Springer, 2005).

[36] For a discussion of Farey Sequences and rational intervals, see Nicholson and Sabat 2020.

[37] Mike Battaglia, private communication, 2024.

[38] For a detailed discussion of "Chord complexity", which includes a closely related approach using a scaled geometric mean measure, see also https://en.xen.wiki/w/Chord_complexity.

proximity to a fundamental, this approach consistently orders sounds in terms of relative harmonicity. Scaling factors prioritising divisibility by lower primes are not enforced, instead methods are suggested to do so *when needed or desired*.

For a single partial $P°$, whether prime or composite, its *harmonic radius* with respect to the fundamental 1° is simply the number P. For any prime number, therefore, its harmonic radius is the same as Euler's degree of *Gradus Suavitas*.

The value $\log_2 P$, which also represents *pitch distance* from 1° to $P°$ measured in octaves, increases as P does. Therefore, it preserves the relative magnitudes of harmonic radius, for any values of P. The base 2 logarithmic harmonic radius, or simply *log2 radius*, of any partial is the same as the Tenney distance of the ratio $1 : P$.

A proportion of two numbers $a : b$ in lowest terms is a set of two natural numbers, which may be interpreted as representing partials $a°$ and $b°$ of their nearest common fundamental, 1°. Each partial may be associated with a vector yielding a point in harmonic space. As noted above, the harmonic distance of the proportion is the length, in octaves, of the shortest direct path between these two partials that passes through the origin. Therefore, it is the sum of the lengths of the two vectors, measured by a base 2 logarithm:

$$\log_2 a + \log_2 b \ = \ \log_2 ab. \tag{14}$$

The log2 radius of the set $\{a°, b°\}$ is defined as the *average* length of the journey from $a°$ to $b°$, passing through the origin, namely, the Tenney distance of $a : b$ divided by 2. The log2 radius is equal to the *arithmetic mean* –

$$\frac{\log_2 a + \log_2 b}{2} = \frac{\log_2 ab}{2} = \log_2 (ab)^{\frac{1}{2}} = \log_2 \sqrt{ab} \tag{15}$$

– and, thus, harmonic radius of the two partials is defined as their *geometric mean*:

$$\sqrt{ab} \tag{16}$$

This value divides the proportion $1 : ab$ into two equal steps, $1 : \sqrt{ab}$ and $\sqrt{ab} : ab$.

For example, if the set contains the partials 2° and 8°, the combined distance, expressed as a proportion, is 16 = (2 · 8) = (4 · 4). Measured as pitch distances in octaves, 2° represents 1 octave, 8° represents 3 octaves, and 16° 4 octaves ($\log_2 16 = 4$), which may also be reached by combining 2 equal steps of 2 octaves (4°). 4 is the geometric mean of 2 and 8; 2 is the arithmetic mean of 1 and 3.

Formalising this, *harmonic radius* is defined as the geometric mean of a set of natural numbers representing partials and *log2 radius* is defined as the arithmetic mean of their pitch distances, measured in octaves.

For an arbitrary set S of partials $\{P°_1, \dots, P°_n\}$,

$$Harmonic\ Radius\ (S) = \sqrt[n]{\prod_i P_i} \tag{17}$$

$$Log_2\ Radius\ (S) = \log_2 \sqrt[n]{\prod_i P_i} = \frac{1}{n}\sum_i \log_2 P_i \tag{18}$$

These quantities represent *average pitch distance from the fundamental partial 1°*, measured geometrically and arithmetically. Note that the partials do not necessarily need to be in lowest terms for this calculation to take place, but to evaluate the relative harmonicity of a set and obtain a unique value for that particular interval or chord, it must first be reduced to lowest terms, i.e., so that its GCD (greatest common divisor) is 1.

One-dimensional harmonic radius of the *pitches* {15°} and {16°} measures their individual pitch distances from the fundamental 1°. Since the GCD of 15 and 16 is 1, two-dimensional harmonic radius of {15°, 16°} measures the diatonic semitone *interval* between partials 15° and 16°, in relation to 1°, their nearest shared fundamental. It averages the two necessary steps: tuning 15° and tuning 16°. The value is $\sqrt{15 \cdot 16} = 4\sqrt{15} \approx 15.49$.

Harmonic radius evaluates partials based on their average magnitudes. However, as Euler's and Barlow's methods recognise, partials with composite numbers may be constructed from smaller, simpler intervallic steps, while prime partials of similar magnitude must be tuned directly. Harmonic radius applied to a set of prime factors or divisors evaluates how large the steps are, on average, but it does not necessarily reflect the size of their product. For example, 15° can be written as a set of prime factors, {3°, 5°}. The harmonic radius of this set is $\sqrt{15}$ or approximately 3.87. Similarly, 16° can be written as a set of its prime factors, {2°, 2°, 2°, 2°}, which produce a harmonic radius value of 2.

To evaluate the degree of divisibility of any number, prime or composite, one possible method is to take the radius of a set containing the number and its greatest prime factor (gpf).[39] Note that this set *is not in lowest terms.* In case P is 1, let

$$gpf(P),\ P \geq 2;\ gpf(1) = 1$$

$$Prime\ Limit\ Radius\ (P) = Harmonic\ Radius\ (\{P, gpf(P)\}). \tag{19}$$

39 Several options have been considered, such as evaluating a partial and its prime factors or its divisors. This method for prime limit radius was chosen because it effectively guarantees a result greater than or equal to the largest prime factor, avoiding (for example) octave transpositions of higher primes obtaining lower values than the primes themselves.

Like Barlow's and Euler's formulae, this equation reduces the harmonic radius of composite numbers by considering the magnitude of their prime factors. The radius of a prime number remains unaltered. See Table 1 below for a comparison of the three orderings applied to the natural numbers.

Sometimes it is musically useful to consider collections of notes as though they were octave-equivalent pitch classes, for example a chord and its family of inversions or when constructing scales and modes that repeat at the octave. In such cases, radius may be computed by ignoring the powers of 2. Let $div2(P)$ represent the pitch-class of a natural number partial P°, calculated by dividing out all powers of 2 until an odd number is obtained. Define

$$Odd\ Radius\ (P) = Harmonic\ Radius\ \big(div2(P)\big). \tag{20}$$

From these expressions and their combination *prime limit odd radius*,[40] pitch sets of arbitrary size may be compared in different ways, depending on what musical information is sought. Radius may be calculated and averaged across all subsets of a given size to compare intervallic, triadic or chordal harmonicities.

Take as an example two dyads, written as {15°, 16°} and {3°, 7°}. The first dyad is a 5-limit diatonic semitone, an interval between two composite number partials, while the second is a 7-limit minor tenth, voiced in its most concordant position as the relationship of two odd primes. In the table below, various measurement algorithms are compared.

Benedetti distance and harmonic radius both express the relative likelihood of perceiving overall harmonicity with respect to a fundamental, which is less likely with two distant partials. On the other hand, the 16:17, 18:19 and 19:20 semitones differ in sound and contextual concordance from the similarly sized lower-prime-limit intervals 15:16 and 20:21. Prime limit radius can reflect these differences, while Benedetti distance and harmonic radius do not: the choice of algorithm depends on the musical context.

For simultaneously sounding dyads, prime limit radius reflects the more concordant sonority of 3:7 but acknowledges the strong saliency of the constituent primes of 15° and 16° – 2, 3, and 5.

In a melodic context, as part of a scale, odd radius provides an accurate comparison of the steps 6:7 (evaluated as 3:7) and 15:16 (evaluated as 15:1), favouring the common diatonic semitone over the septimal minor third. Considering prime limit in the pitch-class context (prime limit odd radius) further increases relative harmonicity in favour of the 5-limit interval. The semitone is easily harmonised as the difference between a major third (4:5) and a perfect fourth (3:4); the more dissonant major seventh 15° does not need to be tuned in one go. On the other hand, the septimal interval necessitates acquiring the sound of partial 7° directly.

40 To calculate the prime limit odd radius of P, calculate the prime limit radius of $div2(P)$.

Both Euler and Barlow values tend to equate the sounds, with Euler slightly favouring the harmonicity of the septimal interval.

	{15°, 16°}		{3°, 7°}	
harmonic radius = √Benedetti distance	{15°, 16°}	15.49	{3°, 7°}	4.58
prime limit radius	{{15°, 5°}, {16°, 2°}}	6.999	{{3°, 3°}, {7°, 7°}}	4.58
odd radius	{15°, 1°}	3.87	{3°, 7°}	4.58
prime limit odd radius	{{15°, 5°}, {1°, 1°}}	2.94	{{3°, 3°}, {7°, 7°}}	4.58
Euler	LCM{15, 16} = 240	11	LCM{3, 7} = 21	9
Barlow*	$\frac{sgn(\xi(16) - \xi(15))}{\xi(15) + \xi(16)}$	–1/13.07	$\frac{sgn(\xi(7) - \xi(3))}{\xi(3) + \xi(7)}$	1/12.95

A similar principle may be observed in the comparison of larger aggregates. Notice that 1 / Tenney intersection and prime limit odd radius obtain nearly identical results.

	major triad {4°, 5°, 6°}		minor triad {10°, 12°, 15°}	
harmonic radius	{4°, 5°, 6°}	4.93	{10°, 12°, 15°}	12.16
prime limit radius	{{4°, 2°}, {5°, 5°}, {6°, 3°}}	3.91	{{10°, 5°}, {12°, 3°}, {15°, 5°}}	7.16
odd radius	{1°, 5°, 3°}	2.47	{5°, 3°, 15°}	6.08
prime limit odd radius	{{1°, 1°}, {5°, 5°}, {3°, 3°}}	2.47	{{5°, 5°}, {3°, 3°}, {15°, 5°}}	5.06
Tenney intersection *	28 coincident partials / 60	1/2.4	12 coincident partials / 60	1/5
Euler	LCM{4, 5, 6} = 60	9	LCM{10, 12, 15} = 60	9
Barlow and Benedetti provide only a pairwise sum of intervals, obtaining the same result for both chords.[41]				

* Barlow's harmonicity measure (in the previous table) and Tenney's intersection are written as fractions, since the values observed in the denominator facilitate comparison between algorithms.

By defining the JI-specific concept of harmonic radius, this paper opens the possibility of evaluating and comparing different collections of harmonic partials. Target intervals or chords may be combined with other contextually relevant partials (largest prime, unique divisors, combination tones, common partial, and others).

41 Euler gives an identical result for *any* chord of any number of notes with LCM 60. The dyad 4:15, a major seventh, the chord 4:5:6:15, a major seventh tetrad, and the harmonic series aggregate 1:2:3:4:5:6:10:12:15:20:30:60 are each assigned the same degree of concordance as the common major triad. Coincidentally, this degree also matches Euler's value for the 3:7 interval in the preceding example.

Such radius-based measures may also be used to compute variously weighted harmonic entropy distributions applicable to non-rationally tuned pitch collections and temperaments. Geometric mean was in fact used as one of the weighting methods in calculating Erlich's harmonic entropy for chords; as such, it already has an active history "as an approximation to the amount of area that 'belongs' to the seed point" (Steve Martin, private communication, 2023).

The final section of this paper presents three tables comparing different harmonic measures, applied to individual numbers, to intervals and finally to triads and tetrads. Brief commentaries of initial observations are sketched, but the practice-based explorations and applications of this research are still in their first stages. The data suggest many possibilities for differentiated approaches to algorithmic composition in JI. Also, since radius may be estimated quite easily by "looking at the numbers", musicians may find here a way of quantifiably comparing many-toned structures tuned in rational intonation while playing and thus be able to invent, shape and discover more finely their unfolding in time as chords and melodies.

6. Tables and Commentaries

Table 1: One-dimensional values (partials ordered with respect to divisibility)
test set: partials 1°–128° | primes: **bold** | early appearances of higher partials: *italic*

EULER Gradus Suavitas	#	BARLOW Indigestibility	#	SABAT Prime Limit Radius	#
1	1°	0	1°	1	1°
2	2°	1	2°	2	2°
3	**3°**	2	4°	2.828	4°
3	4°	2.667	**3°**	3	**3°**
4	6°	3	8°	4	8°
4	8°	3.667	6°	4.243	6°
5	**5°**	4	16°	5	**5°**
5	9°	4.667	12°	5.196	9°
5	12°	5	32°	5.657	16°
5	16°	5.333	9°	6	12°
6	10°	5.667	24°	7	**7°**
6	18°	6	*64°*	7.071	10°
6	24°	6.333	18°	7.348	18°
6	32°	6.4	**5°**	8	32°
7	**7°**	6.667	48°	8.485	24°
7	15°	7	*128°*	8.66	15°
7	20°	7.333	36°	9	27°
7	27°	7.4	10°	9.899	14°
7	36°	7.667	96°	10	20°

EULER Gradus Suavitas	#	BARLOW Indigestibility	#	SABAT Prime Limit Radius	#
7	48°	8	27°	10.392	36°
7	64°	8.333	72°	11	**11°**
8	14°	8.4	20°	11.18	25°
8	30°	9	*54°*	11.314	64°
8	40°	9.067	15°	12	48°
8	54°	9.4	40°	12.124	21°
8	72°	10	*108°*	12.247	30°
8	96°	10.067	30°	12.728	54°
8	128°	10.286	**7°**	13	**13°**
9	*21°*	10.4	80°	14	28°
9	25°	10.667	*81°*	14.142	40°
9	28°	11.067	60°	14.697	72°
9	*45°*	11.286	14°	15	45°
9	60°	11.733	*45°*	15.556	22°
9	80°	12.067	120°	15.588	81°
9	*81°*	12.286	28°	15.652	35°
9	108°	12.733	*90°*	15.811	50°
10	*42°*	12.8	25°	16	128°
10	50°	12.952	21°	16.971	96°
10	56°	13.286	56°	17	**17°**
10	*90°*	13.8	50°	17.146	42°
10	120°	13.952	42°	17.321	60°
11	**11°**	14.286	112°	18	108°
11	35°	14.8	*100°*	18.385	26°
11	63°	14.952	84°	18.52	49°
11	75°	15.467	75°	19	**19°**
11	84°	15.619	63°	19.053	33°
11	100°	16.619	*126°*	19.365	75°
11	112°	16.686	35°	19.799	56°
12	22°	17.686	*70°*	20	80°
12	*70°*	18.182	**11°**	21	63°
12	*126°*	19.182	22°	21.213	90°
13	**13°**	19.2	*125°*	22	44°
13	33°	19.352	*105°*	22.136	70°
13	44°	20.182	44°	22.361	100°
13	49°	20.571	49°	22.517	39°
13	105°	20.848	*33°*	23	**23°**
13	125°	21.182	88°	24.042	34°
14	26°	21.571	*98°*	24.249	84°
14	66°	21.848	*66°*	24.495	120°
14	88°	22.154	**13°**	24.597	55°

<u>Commentary on Table 1:</u>

This list shows one-dimensional rankings according to three measures: Euler, Barlow, and Sabat. To compare the respective ordering of partials, weigh how readily the step(s) from fundamental to partial can be imagined against how many steps need to be taken. The *prime limit radius* measure is used because it distinguishes the divisibility of numbers, whereas *harmonic radius* does not. The first occurrences of primes are marked in boldface and highlighted yellow; some unexpectedly early appearances of higher partials that require several steps are marked in italic and highlighted pink.

Barlow *indigestibility* introduces higher primes more slowly than Euler *GS*. Considerably higher composite numbers (marked in italics and highlighted) precede smaller prime number intervals (like 1:5 or 1:7), which may be easily perceived and tuned. Barlow reaches 64° (six octaves, the span from the lowest contrabass E to the high E at the end of the violin fingerboard) before including 5° and includes partials from the seventh octave before reaching 7°. By the time Barlow reaches 13°, *prime limit radius* has already included all primes up to and including 23° – the highest prime that is directly tuneable according to Sabat's 2005 list of tuneable intervals.

Euler's list is more inclusive towards primes, but still favours less salient higher partials (81°, 90°) before including the tuneable quartertones 11° and 13°. Euler's integer-valued degree categories do not allow for a finer quantitative differentiation possible with the Barlow and Sabat measures. Partials sharing the same *Gradus Suavitas* are simply sorted in increasing order.

<u>Table 2: Two-dimensional values (intervals)</u>

search constrained between 8:9 and 1:8, partials to 28°; tuneable intervals are marked "T"

primes: **bold** | easily tuned by means of third note: <u>underlined</u> |

early appearance of difficult-to-tune intervals: *italic*

EULER	ratio		cents	BARLOW	ratio			SABAT	ratio		cents	SABAT	ratio		cents
2	1:2	T	1200	1	1:2	T	1200	1.414	1:2	T	1200	1.414	1:2	T	1200
3	**1:3**	T	1901.955	2	1:4	T	2400	1.732	**1:3**	T	1901.955	1.732	**1:3**	T	1901.955
3	1:4	T	2400	2.667	**1:3**	T	1901.955	2	1:4	T	2400	1.834	1:4	T	2400
4	1:6	T	3101.955	3	1:8	T	3600	2.236	**1:5**	T	2786.314	2.236	**1:5**	T	2786.314
4	1:8	T	3600	3.667	1:6	T	3101.955	2.449	1:6	T	3101.955	2.246	1:6	T	3101.955
4	2:3	T	701.955	3.667	2:3	T	701.955	2.449	2:3	T	701.955	2.378	1:8	T	3600
5	**1:5**	T	2786.314	-4.667	3:4	T	498.045	2.646	**1:7**	T	3368.826	2.449	2:3	T	701.955
5	3:4	T	498.045	5.667	3:8	T	1698.045	2.828	1:8	T	3600	2.646	**1:7**	T	3368.826
6	2:5	T	1586.314	6.333	2:9	T	2603.91	3.162	2:5	T	1586.314	3.162	2:5	T	1586.314
6	2:9	T	2603.91	6.4	**1:5**	T	2786.314	3.464	3:4	T	498.045	3.177	3:4	T	498.045
6	3:8	T	1698.045	6.667	3:16	T	2898.045	3.742	2:7	T	2168.826	3.698	2:9	T	2603.91
7	**1:7**	T	3368.826	7.333	4:9	T	1403.91	3.873	3:5	T	884.359	3.742	2:7	T	2168.826

EULER	ratio		cents	BARLOW	ratio			SABAT	ratio		cents	SABAT	ratio		cents
7	3:5	T	884.359	7.4	2:5	T	1586.314	4.243	2:9	T	2603.91	3.873	3:5	T	884.359
7	3:16	T	2898.045	8.333	8:9		203.91	4.472	4:5	T	386.314	4.101	4:5	T	386.314
7	4:5	T	386.314	8.4	4:5	T	386.314	4.583	3:7	T	1466.871	4.12	3:8	T	1698.045
7	4:9	T	1403.91	9.067	3:5	T	884.359	4.69	**2:11**	T	2951.318	4.583	3:7	T	1466.871
8	2:7	T	2168.826	-9.333	9:16		996.09	4.899	3:8	T	1698.045	4.69	**2:11**	T	2951.318
8	2:15	T	3488.269	-9.4	5:8	T	813.686	5.099	**2:13**	T	3240.528	4.774	2:15	T	3488.269
8	3:10	T	2084.359	10	4:27	T	3305.865	5.292	4:7	T	968.826	4.796	4:9	T	1403.91
8	5:6	T	315.641	10.067	2:15	T	3488.269	5.477	3:10	T	2084.359	4.852	4:7	T	968.826
8	5:8	T	813.686	10.067	3:10	T	2084.359	5.477	5:6	T	315.641	5.023	3:10	T	2084.359
8	8:9		203.91	-10.067	5:6	T	315.641	5.477	2:15	T	3488.269	5.023	5:6	T	315.641
9	3:7	T	1466.871	10.286	**1:7**	T	3368.826	5.745	3:11	T	2249.363	5.099	**2:13**	T	3240.528
9	3:20	T	3284.359	-10.4	5:16	T	2013.686	5.916	5:7	T	582.512	5.318	5:8	T	813.686
9	4:7	T	968.826	11	8:27	T	2105.865	6	4:9	T	1403.91	5.342	3:16	T	2898.045
9	4:15	T	2288.269	11.067	3:20	T	3284.359	6.245	3:13	T	2538.573	5.745	3:11	T	2249.363
9	4:27	T	3305.865	11.067	4:15	T	2288.269	6.325	5:8	T	813.686	5.847	5:9	T	1017.596
9	5:9	T	1017.596	-11.067	5:12	T	1515.641	6.481	3:14	T	2666.871	5.916	5:7	T	582.512
9	5:12	T	1515.641	11.286	2:7	T	2168.826	6.481	6:7	T	266.871	5.943	3:14	T	2666.871
9	5:16	T	2013.686	-11.733	5:9	T	1017.596	6.633	4:11	T	1751.318	5.943	6:7	T	266.871
9	9:16		996.09	12	*16:27*		905.865	6.708	5:9	T	1017.596	6.083	4:11	T	1751.318
10	3:14	T	2666.871	-12.067	5:24	T	2715.641	6.928	3:16	T	2898.045	6.192	4:15	T	2288.269
10	5:18	T	2217.596	12.067	8:15	T	1088.269	7.141	**3:17**	T	3003	6.22	8:9		203.91
10	5:24	T	2715.641	12.286	4:7	T	968.826	7.211	4:13	T	2040.528	6.245	3:13	T	2538.573
10	6:7	T	266.871	-12.733	5:18	T	2217.596	7.416	5:11	T	1365.004	6.293	7:8	T	231.174
10	7:8	T	231.174	12.952	3:7	T	1466.871	7.483	7:8	T	231.174	6.514	5:12	T	1515.641
10	8:15	T	1088.269	-13.286	7:8	T	231.174	7.55	**3:19**	T	3195.558	6.514	3:20	T	3284.359
10	8:27	T	2105.865	13.733	9:20		1382.404	7.746	5:12	T	1515.641	6.613	4:13	T	2040.528
11	4:21	T	2870.781	13.952	3:14	T	2666.871	7.746	3:20	T	3284.359	6.897	5:16	T	2013.686
11	4:25	T	3172.627	13.952	6:7	T	266.871	7.746	4:15	T	2288.269	6.919	7:9	T	435.084
11	5:7	T	582.512	-14.286	7:16		1431.174	7.937	7:9	T	435.084	7.141	**3:17**	T	3003
11	5:27		2919.551	14.4	5:27		2919.551	8.062	5:13	T	1654.214	7.241	4:27	T	3305.865
11	7:9	T	435.084	14.8	4:25	T	3172.627	8.124	3:22	T	3449.363	7.326	4:21	T	2870.781
11	7:12	T	933.129	14.952	4:21	T	2870.781	8.124	6:11	T	1049.363	7.416	5:11	T	1365.004
11	7:16		1431.174	-14.952	7:12	T	933.129	8.246	4:17	T	2504.955	7.45	3:22	T	3449.363
11	9:20		1382.404	15.4	10:27		1719.551	8.307	**3:23**	T	3526.319	7.45	6:11	T	1049.363
11	*16:27*		905.865	-15.619	7:9	T	435.084	8.367	5:14	T	1782.512	7.499	4:25	T	3172.627
12	**2:11**	T	2951.318	15.8	8:25	T	1972.627	8.367	7:10	T	617.488	7.55	**3:19**	T	3195.558
12	5:14	T	1782.512	-15.952	7:24	T	2133.129	8.485	8:9		203.91	7.562	4:17	T	2504.955
12	6:25	T	2470.672	15.952	8:21		1670.781	8.718	4:19	T	2697.513	7.583	5:18	T	2217.596
12	7:10	T	617.488	-16.4	*20:27*		519.551	8.775	7:11	T	782.492	7.672	5:14	T	1782.512
12	7:18	T	1635.084	16.467	6:25	T	2470.672	8.832	6:13	T	1338.573	7.672	7:10	T	617.488
12	7:24	T	2133.129	-16.619	7:18	T	1635.084	8.944	5:16	T	2013.686	7.707	7:12	T	933.129
12	8:21		1670.781	16.619	9:14	T	764.916	9.165	7:12	T	933.129	7.888	8:11	T	551.318

EULER	ratio		cents	BARLOW	ratio		cents	SABAT	ratio		cents	SABAT	ratio		cents
12	8:25	T	1972.627	16.686	5:7	T	582.512	9.165	4:21	T	2870.781	7.994	4:19	T	2697.513
12	9:14	T	764.916	16.8	16:25		772.627	9.22	5:17	T	2118.642	8.03	8:15	T	1088.269
12	10:27		1719.551	16.952	*16:21*		470.781	9.381	8:11	T	551.318	8.062	5:13	T	1654.214
13	3:11	T	2249.363	17.467	12:25		1270.672	9.487	5:18	T	2217.596	8.066	9:16		996.09
13	4:11	T	1751.318	17.619	9:28	T	1964.916	9.539	7:13	T	1071.702	8.099	6:13	T	1338.573
13	5:21	T	2484.467	17.686	5:14	T	1782.512	9.592	4:23	T	3028.274	8.161	7:16		1431.174
13	5:28	T	2982.512	-17.686	7:10	T	617.488	9.747	5:19	T	2311.199	8.307	**3:23**	T	3526.319
13	7:15		1319.443	18.133	9:25		1768.717	9.95	9:11	T	347.408	8.447	5:24	T	2715.641
13	7:20	T	1817.488	-18.286	7:27	T	2337.039	10	4:25	T	3172.627	8.575	8:13	T	840.528
13	7:27	T	2337.039	18.686	5:28	T	2982.512	10.1	6:17	T	1803	8.673	9:11	T	347.408
13	9:25		1768.717	-18.686	7:20	T	1817.488	10.198	8:13	T	840.528	8.775	7:11	T	782.492
13	9:28	T	1964.916	19.133	18:25		568.717	10.247	5:21	T	2484.467	8.796	4:23	T	3028.274
13	12:25		1270.672	19.182	**2:11**	T	2951.318	10.247	7:15		1319.443	8.828	5:27		2919.551
13	*16:21*		470.781	-19.286	14:27		1137.039	10.392	4:27	T	3305.865	8.932	5:21	T	2484.467
13	16:25		772.627	19.352	5:21	T	2484.467	10.488	5:22	T	2565.004	8.932	7:15		1319.443
13	*20:27*		519.551	-19.352	7:15		1319.443	10.583	7:16		1431.174	8.973	7:18	T	1635.084
14	**2:13**	T	3240.528	20.182	4:11	T	1751.318	10.677	6:19	T	1995.558	8.973	9:14	T	764.916
14	3:22	T	3449.363	20.352	10:21		1284.467	10.724	5:23	T	2641.961	9.184	6:25	T	2470.672
14	6:11	T	1049.363	20.848	3:11	T	2249.363	10.817	9:13	T	636.618	9.22	5:17	T	2118.642
14	8:11	T	551.318	21.182	8:11	T	551.318	10.909	7:17	T	1536.13	9.261	6:17	T	1803
14	10:21		1284.467	21.352	15:28		1080.557	10.954	5:24	T	2715.641	9.391	8:27	T	2105.865
14	14:27		1137.039	21.848	3:22	T	3449.363	10.954	8:15	T	1088.269	9.429	9:13	T	636.618
14	18:25		568.717	21.848	6:11	T	1049.363	11.225	7:18	T	1635.084	9.501	8:21		1670.781
15	3:13	T	2538.573	-22.182	11:16	T	648.682	11.225	9:14	T	764.916	9.539	7:13	T	1071.702
15	4:13	T	2040.528	23.086	*7:25*		2203.802	11.402	5:26	T	2854.214	9.618	5:22	T	2565.004
15	5:11	T	1365.004	23.154	**2:13**	T	3240.528	11.402	10:13	T	454.214	9.725	8:25	T	1972.627
15	*7:25*		2203.802	23.515	9:11	T	347.408	11.533	7:19	???	1728.687	9.747	5:19	T	2311.199
15	9:11	T	347.408	-23.848	11:24		1350.637	11.619	5:27		2919.551	9.791	6:19	T	1995.558
15	11:16	T	648.682	24.086	14:25		1003.802	11.662	8:17		1304.955	9.806	8:17		1304.955
15	15:28		1080.557	24.154	4:13	T	2040.528	11.747	6:23	T	2326.319	9.834	9:20		1382.404
16	5:22	T	2565.004	24.515	*9:22*		1547.408	11.832	5:28	T	2982.512	9.95	5:28	T	2982.512
16	6:13	T	1338.573	-24.515	11:18		852.592	11.832	7:20	T	1817.488	9.95	7:20	T	1817.488
16	8:13	T	840.528	24.582	5:11	T	1365.004	11.958	*11:13*		289.21	9.995	7:24	T	2133.129
16	*9:22*		1547.408	24.821	3:13	T	2538.573	12	9:16		996.09	10.23	11:16	T	648.682
16	11:18		852.592	25.154	8:13	T	840.528	12.247	6:25	T	2470.672	10.367	8:19	T	1497.513
16	11:24		1350.637	25.582	5:22	T	2565.004	12.329	8:19	T	1497.513	10.446	7:27	T	2337.039
16	14:25		1003.802	-25.752	21:25		301.847	12.369	9:17		1101.045	10.455	5:26	T	2854.214
17	5:13	T	1654.214	25.821	6:13	T	1338.573	12.41	7:22	T	1982.492	10.455	10:13	T	454.214
17	7:11	T	782.492	-26.154	13:16		359.472	12.41	11:14		417.508	10.693	9:25	???	1768.717
17	9:13	T	636.618	-26.182	11:27		1554.547	12.689	*7:23*		2059.448	10.724	5:23	T	2641.961
17	*11:15*		536.951	-26.582	11:20		1034.996	12.845	*11:15*		536.951	10.772	6:23	T	2326.319
17	11:20		1034.996	-27.182	22:27		354.547	12.961	7:24	T	2133.129	10.782	9:17		1101.045

EULER	ratio		cents	BARLOW	ratio			SABAT	ratio		cents	SABAT	ratio		cents
17	11:27		1554.547	-27.248	*11:15*		536.951	12.961	8:21		1670.781	10.818	*7:25*		2203.802
17	13:16		359.472	27.487	9:13	T	636.618	13.038	10:17		918.642	10.909	7:17	T	1536.13
17	21:25		301.847	-27.821	13:24		1061.427	13.077	9:19		1293.603	11.121	13:16		359.472
18	5:26	T	2854.214	28.248	15:22		663.049	13.229	*7:25*		2203.802	11.197	*11:15*		536.951
18	7:22	T	1982.492	28.468	7:11	T	782.492	13.266	11:16	T	648.682	11.248	*9:22*		1547.408
18	*9:26*		1836.618	28.487	*9:26*		1836.618	13.416	9:20		1382.404	11.248	11:18		852.592
18	10:13	T	454.214	-28.487	13:18		563.382	13.491	7:26		2271.702	11.38	7:22	T	1982.492
18	11:14		417.508	28.554	5:13	T	1654.214	13.565	8:23	T	1828.274	11.38	11:14		417.508
18	13:18		563.382	29.468	7:22	T	1982.492	13.675	*11:17*		753.637	11.399	9:19		1293.603
18	13:24		1061.427	-29.468	11:14		417.508	13.748	7:27	T	2337.039	11.406	8:23	T	1828.274
18	15:22		663.049	29.554	5:26	T	2854.214	13.784	10:19		1111.199	11.449	10:27		1719.551
18	22:27		354.547	29.554	10:13	T	454.214	13.964	13:15		247.741	11.533	7:19		1728.687
19	**3:17**	T	3003	-30.154	13:27		1265.337	14.071	*9:22*		1547.408	11.584	10:21		1284.467
19	4:17	T	2504.955	-30.468	11:28		1617.508	14.071	11:18		852.592	11.636	9:28	T	1964.916
19	7:13	T	1071.702	-30.554	13:20		745.786	14.142	8:25	T	1972.627	11.911	12:25		1270.672
19	11:21		1119.463	-30.982	11:25		1421.309	14.283	12:17		603	11.956	10:17		918.642
19	11:25		1421.309	-31.134	11:21		1119.463	14.387	9:23		1624.364	11.958	*11:13*		289.21
19	11:28		1617.508	-31.221	13:15		247.741	14.422	13:16		359.472	12.01	12:17		603
19	13:15		247.741	-31.982	22:25		221.309	14.457	11:19		946.195	12.172	13:15		247.741
19	13:20		745.786	32.118	**4:17**	T	2504.955	14.491	10:21		1284.467	12.178	*16:27*		905.865
19	13:27		1265.337	32.221	15:26		952.259	14.697	8:27	T	2105.865	12.228	13:18		563.382
20	6:17	T	1803	32.44	7:13	T	1071.702	14.832	11:20		1034.996	12.228	*9:26*		1836.618
20	7:26		2271.702	32.784	3:17	T	3003	14.866	13:17		464.428	12.321	*16:21*		470.781
20	8:17		1304.955	33.118	8:17		1304.955	15	9:25		1768.717	12.371	7:26		2271.702
20	15:26		952.259	33.44	7:26		2271.702	15.1	12:19		795.558	12.473	11:20		1034.996
20	22:25		221.309	33.784	6:17	T	1803	15.166	10:23		1441.961	12.529	11:24		1350.637
21	**3:19**	T	3195.558	-34.44	13:28		1328.298	15.199	11:21		1119.463	12.541	9:23		1624.364
21	4:19	T	2697.513	34.784	12:17		603	15.297	*9:26*		1836.618	12.612	16:25		772.627
21	5:17	T	2118.642	-34.954	13:25		1132.1	15.297	13:18		563.382	12.64	10:19		1111.199
21	9:17		1101.045	-35.106	13:21		830.253	15.427	14:17		336.13	12.689	*7:23*		2059.448
21	12:17		603	35.451	9:17		1101.045	15.716	13:19		656.985	12.697	12:19		795.558
21	13:21		830.253	-35.784	17:24		597	15.875	9:28	T	1964.916	13.095	11:27		1554.547
21	13:25		1132.1	36.105	**4:19**	T	2697.513	15.906	11:23		1276.956	13.248	11:21		1119.463
21	13:28		1328.298	36.106	21:26		369.747	15.969	15:17		216.687	13.445	16:19		297.513
22	6:19	T	1995.558	36.518	5:17	T	2118.642	16.125	13:20		745.786	13.547	14:27		1137.039
22	8:19	T	1497.513	36.772	3:19	T	3195.558	16.248	11:24		1350.637	13.559	13:20		745.786
22	10:17		918.642	37.105	8:19	T	1497.513	16.31	14:19		528.687	13.561	11:25		1421.309
22	17:24		597	37.518	10:17		918.642	16.432	10:27		1719.551	13.62	13:24		1061.427
22	21:26		369.747	37.772	6:19	T	1995.558	16.523	13:21		830.253	13.675	*11:17*		753.637
23	5:19	T	2311.199	38.105	16:19		297.513	16.583	11:25		1421.309	13.866	18:25		568.717
23	7:17	T	1536.13	-38.118	17:27		800.91	16.613	12:23	T	1126.319	13.907	10:23		1441.961
23	9:19		1293.603	-38.518	17:20		281.358	16.882	15:19		409.244	13.92	15:17		216.687
23	*11:13*		289.21	38.772	12:19		795.558	16.912	11:26		1489.21	13.97	12:23	T	1126.319

Commentary on Table 2:

This list shows two-dimensional rankings according to four measures: Euler, Barlow, and two variations of harmonic radius. As before, the first occurrences of primes are marked in boldface and highlighted yellow; unexpectedly early appearances of difficult-to-tune intervals are marked in italic and highlighted pink. Ratios marked in green are not tuneable *directly* (as simultaneous dyads) but may be relatively easily tuned with the addition of a third note – e.g., 9:16 with the addition of 12 – or, by comparison, alternation and/or superposition with a complementary dyad that produces the upper note as a summation tone – e.g., 8:15 and 7:15 or 8:17 and 9:17. A double line in both radius measures indicates the point at which intervals that are neither directly nor indirectly tuneable begin to be interspersed in the list. A dashed line indicates the boundary introducing difficult-to-tune intervals.

The tuneable interval list cited in this paper (see Appendix 1 below) was originally conceived as a tool for composing in rational intonation, since musicians playing strings or other freely pitched acoustic instruments can exactly tune ratios of numbers with smaller magnitudes by ear. Such ratios are logical material for composing music in JI, since their "in-tune-ness" highlights specifically rational tonal interactions. A list of lowest terms ratios from natural numbers up to 28 was ordered by increasing pitch distance and assessed for tuneability in various registers using solo and duo instrumentations. Some intervals could be tuned reliably and others not.

As Table 2 demonstrates, the list correlates well with harmonic radius. Sometimes it was possible to recognise beating at the common partial; sometimes periodicity and stable phase of the interval was key; in other cases, the reinforcement of combination tones provided needed cues. Intervals smaller than 8:9 were generally not tuneable due to critical band effects (roughness, amplitude modulation, difficulty of frequency discrimination). Intervals wider than 1:8 (three octaves) were ignored, although some large tuneable intervals *do exist* (e.g., 1:9, 1:10, ...). Low register intervals, for which the fundamental lies below 20 Hz, are not readily tuneable by phase/periodicity information, but if the common partial is clearly heard, beating may serve as a cue.

For intervals *below* a stable pitch, the range from A 440 Hz down to A 55 Hz was assessed. Intervals *above* were assessed from A 220 Hz up to A 1760 Hz. The list of intervals has not been studied formally with a large test sample of listeners, but its utility has been verified in the context of practice-based research. These intervals serve as structural points of reference in numerous compositions by the author and various colleagues (Wolfgang von Schweinitz, Thomas Nicholson, Juhani Nuorvala, and others). A version of the list is appended to this paper for reference.

As noted above, radius produces the same ordering of intervals as Tenney/Benedetti. The second version of harmonic radius is a combination of *prime limit radius* with the LCM. This balances information relevant to sounding simultaneous dyads (saliency of common partial) with divisibility, which sometimes favours several smaller steps (simpler building blocks).

This measurement gives a remarkably well-correlated result with the tuneable interval list through 10:13 (indicated with a double-line border on all four measures for comparison). Keeping the second radius value under 11 (dashed border) produces a list with all but four of the tuneable dyads and no unexpected discordances. The first radius measure has a dashed line at the point where tuneable sounds are increasingly interspersed with less salient ones.

Euler's measure correlates well in the beginning, but several dyads that are not tuneable, like 16:27, 16:21 and 20:27, occur relatively early in the list. They are seen even earlier in Barlow's list. While such intervals may be common in 3-, 5-, and 7-limit scales, they are less likely to be sounded simultaneously and are not directly tuneable.

Table 3: Triads and tetrads

unique odd partials to 27° – pitch-classes, no unisons | primes: **bold** | chords shaded by prime limit (grey = 3°/17°/19°; ivory = 5°; pink = 7°; green = 11°; purple=13°)

TENNEY Intersection		SABAT Odd Harmonic Radius		Prime Limit Odd Radius		Prime Limit Odd Radius + LCM		chord
1	**1:3:5**	2.466212074	**1:3:5**	2.466212074	**1:3:5**	2.68787538	1:3:9	tertial triad
1	**1:3:7**	2.758924176	**1:3:7**	2.498049533	1:3:9	2.954176939	**1:3:5**	quintal major triad
1	1:3:9	3	1:3:9	2.758924176	**1:3:7**	3.297225342	1:3:15	
1	**1:3:11**	3.201085873	1:3:5:7	2.961765219	1:3:15	3.379774445	**1:3:7**	septimal triad 6:7:8
1	**1:3:13**	3.20753433	**1:3:11**	2.961765219	1:5:9	3.534154448	1:3:5:15	quintal major seventh tetrad
1	1:3:15	3.27106631	1:5:7	2.971277986	1:3:5:9	3.651895815	1:5:15	
1	**1:3:17**	3.391211443	**1:3:13**	3	1:3:27	3.737192819	1:3:27	
1	**1:3:19**	3.408658099	1:3:5:9	3.201085873	1:3:5:7	3.772244581	1:3:21	septimal sus4 triad 16:21:24
1	1:3:21	3.556893304	1:5:9	3.20753433	**1:3:11**	3.897902238	1:3:5:9	quintal major ninth tetrad
1	**1:3:23**	3.556893304	1:3:15	3.22496808	1:3:25	3.948222039	1:3:9:27	tertial tetrad (open strings)
1	1:3:25	3.584024634	1:3:5:11	3.232029337	1:3:7:9	4.049538907	**1:3:11**	undecimal triad 8:11:12
1	1:3:27	3.707792751	1:3:7:9	3.27106631	1:5:7	4.107455623	1:5:9	
1	1:5:7	3.708429769	**1:3:17**	3.313294	1:3:21	4.145980143	1:5:7	
1	1:5:9	3.736875706	1:3:5:13	3.313294	1:7:9	4.171167511	1:9:27	
1	1:5:11	3.802952461	1:5:11	3.376023591	1:3:5:15	4.181667936	1:3:7:21	
1	1:5:13	3.848501131	**1:3:19**	3.391211443	**1:3:13**	4.256699613	1:5:25	quintal augmented triad
1	1:5:15	3.872983346	1:3:5:15	3.408658099	1:3:5:27	4.271604615	1:3:9:15	
1	1:5:17	3.898548981	1:3:7:11	3.408658099	1:3:9:15	4.329381358	**1:3:13**	tridecimal triad 8:12:13
1	1:5:19	3.979057208	1:7:9	3.441608071	1:3:9:27	4.360540131	1:3:7:9	septimal tetrad 6:7:8:9
1	1:5:21	3.979057208	1:3:21	3.511560959	1:5:15	4.468844062	1:7:21	

TENNEY Intersection		SABAT Odd Harmonic Radius		Prime Limit Odd Radius		Prime Limit Odd Radius + LCM		chord
1	1:5:23	3.996088015	1:3:5:17	3.556893304	1:5:27	4.549279605	3:5:15	quintal minor chord 10:12:15
1	1:5:25	4.020725759	1:5:13	3.556893304	1:9:15	4.584426407	1:9:15	
1	1:5:27	4.064813851	1:3:7:13	3.584024634	1:3:5:11	4.651208827	1:5:9:15	
1	1:7:9	4.10156593	**1:3:23**	3.598624782	1:3:5:25	4.699201787	1:7:9	septimal triad 7:8:9
1	1:7:11	4.108764171	1:3:5:19	3.602810866	1:9:27	4.71769398	1:3:5:7	quintal-septimal tetrad 4:5:6:7
1	1:7:13	4.151347726	1:3:9:11	3.618669796	1:3:9:11	4.778596848	1:3:9:21	
1	1:7:15	4.212865931	1:3:5:21	3.633411077	1:3:9:25	4.787706344	1:3:25	
1	1:7:17	4.212865931	1:3:7:15	3.672294326	1:3:5:21	4.819789491	**1:3:17**	
1	1:7:19	4.212865931	1:5:7:9	3.672294326	1:3:7:15	4.822452087	1:3:5:25	
1	1:7:21	4.217163327	1:3:25	3.672294326	1:5:7:9	4.967582355	1:5:11	undecimal triad 8:10:11
1	1:7:23	4.217163327	1:5:15	3.707792751	1:3:7:27	5.039064789	**1:3:19**	nodecimal triad 3:16:19
1	1:7:25	4.254320865	1:7:11	3.707792751	1:3:9:21	5.069578311	1:3:9:11	undecimal tetrad 8:9:11:12
1	1:7:27	4.309776748	1:3:5:23	3.708429769	**1:3:17**	5.116784553	3:5:9	
1	1:9:11	4.326748711	1:3:27	3.736875706	1:3:5:13	5.12992784	1:3:5:27	quintal added sixth tetrad 16:20:24:27
1	1:9:13	4.328393928	1:3:9:13	3.772998411	1:3:9:13	5.169991998	1:3:5:21	
1	1:9:15	4.346773933	1:3:7:17	3.802952461	1:5:11	5.169991998	1:3:7:15	
1	1:9:17	4.396829672	1:5:17	3.823622457	1:5:25	5.244888014	1:9:21	
1	1:9:19	4.400558684	1:3:5:25	3.848501131	**1:3:19**	5.284793547	1:3:15:25	
1	1:9:21	4.429606853	1:5:7:11	3.852046512	1:9:11	5.310865987	1:5:13	tridecimal augmented triad 8:10:13
1	1:9:23	4.469338246	1:3:7:19	3.872983346	1:9:25	5.359884014	1:3:9:13	tridecimal tetrad 4:6:9:13
1	1:9:25	4.486046344	1:3:5:27	3.872983346	1:5:9:15	5.439257714	**1:3:23**	
1	1:9:27	4.486046344	1:3:9:15	3.898548981	1:3:7:11	5.484806552	1:3:5:11	undecimal tetrad 8:10:11:12
1	1:11:13	4.497941445	1:7:13	3.910421742	1:3:15:27	5.503384502	1:7:9:21	
1	1:11:15	4.551078463	1:3:11:13	3.910421742	1:5:9:27	5.566977312	3:7:21	subharmonic septimal triad 21:24:28
1	1:11:17	4.562902635	1:5:19	3.914430398	1:3:7:25	5.585808567	3:5:9:15	quintal minor seventh tetrad
1	1:11:19	4.582575695	1:3:7:21	3.928344415	1:5:21	5.621747827	1:3:15:27	
1	1:11:21	4.618520218	1:5:7:13	3.928344415	1:7:15	5.621747827	1:5:9:27	
1	1:11:23	4.626065009	1:9:11	3.979057208	1:7:27	5.629433102	1:5:7:15	
1	1:11:25	4.628637519	1:3:9:17	3.979057208	1:9:21	5.630435041	1:9:11	undecimal triad 8:9:11
1	1:11:27	4.687991145	1:3:7:23	3.994564982	1:3:7:21	5.665653044	1:3:15:21	
1	1:13:15	4.716841683	1:5:9:11	3.996088015	1:3:5:17	5.683243842	1:7:11	
1	1:13:17	4.716841683	1:3:11:15	4.020725759	1:5:13	5.710961816	1:5:27	
1	1:13:19	4.71769398	1:5:21	4.034716409	1:3:9:17	5.738793548	1:3:7:27	
1	1:13:21	4.71769398	1:7:15	4.064813851	1:3:7:13	5.754436707	1:5:15:25	
1	1:13:23	4.71769398	3:5:7	4.072631145	1:9:13	5.764525891	1:5:21	

TENNEY Intersection		SABAT Odd Harmonic Radius		Prime Limit Odd Radius		Prime Limit Odd Radius + LCM		chord
1	1:13:25	4.759149431	1:3:9:19	4.10156593	**1:3:23**	5.764525891	1:7:15	
1	1:13:27	4.786739859	1:3:7:25	4.108764171	1:3:5:19	5.798889998	1:3:5:13	tridecimal tetrad 8:10:12:13
1	1:15:17	4.786739859	1:5:7:15	4.111602703	1:3:11:15	5.82872336	1:3:9:25	
1	1:15:19	4.862944131	1:5:23	4.111602703	1:5:9:11	5.853941057	3:7:9	septimal minor triad 6:7:9
1	1:15:21	4.866768801	1:3:11:17	4.128352032	1:3:15:25	5.861254531	1:3:9:17	Dreams of China tetrad LM Young 6:8:9:17
1	1:15:23	4.879729685	1:3:7:27	4.128352032	1:5:9:25	5.912451215	1:5:17	
1	1:15:25	4.879729685	1:3:9:21	4.148481755	1:3:9:19	5.918469043	1:15:25	
1	1:15:27	4.890973247	1:9:13	4.151347726	1:3:11:27	5.954144019	1:5:7:21	
1	1:17:19	4.918005007	1:5:9:13	4.168258963	1:3:25:27	6.010649717	1:3:11:15	
1	1:17:21	4.918005007	1:3:13:15	4.172531932	1:5:7:15	6.019524955	1:9:13	tridecimal triad 4:9:13
1	1:17:23	4.918684734	1:7:17	4.212865931	1:3:15:21	6.075983096	1:7:13	tridecimal triad 4:7:13
1	1:17:25	4.938888725	1:5:7:17	4.212865931	1:5:7:27	6.082640389	1:3:9:19	nodecimal minor ninth tetrad 6:16:18:19
1	1:17:27	4.991980728	1:3:9:23	4.212865931	1:5:9:21	6.121335066	1:5:15:27	
1	1:19:21	5	1:5:25	4.212865931	1:7:9:15	6.13579244	1:3:7:11	
1	1:19:23	5.003995209	1:3:11:19	4.25358982	1:3:21:27	6.16071992	1:9:15:27	
1	1:19:25	5.0743262	1:3:13:17	4.25358982	1:7:9:27	6.169142003	1:5:15:21	
1	1:19:27	5.07814867	1:5:7:19	4.254320865	1:7:11	6.181436925	1:5:19	nodecimal major-minor triad 4:10:19
1	1:21:23	5.097132735	1:3:9:25	4.271604615	1:15:27	6.208834449	1:5:7:9	
1	1:21:25	5.097132735	1:5:9:15	4.271604615	3:5:9	6.28898713	1:3:21:27	
1	1:21:27	5.104468722	1:7:19	4.277444161	1:7:25	6.28898713	1:7:9:27	
1	1:23:25	5.12992784	1:5:27	4.286953864	1:3:13:15	6.325269095	5:9:15	
1	1:23:27	5.12992784	3:5:9	4.286953864	1:5:9:13	6.341325705	1:3:5:17	septdecimal flat ninth 8:10:12:17
1	1:25:27	5.12992784	1:9:15	4.309776748	1:3:5:23	6.346703871	1:5:9:25	
1	1:3:5:7	5.130779001	1:3:11:21	4.328393928	1:3:13:27	6.354845187	1:3:13:15	tridecimal tetrad 4:12:13:15
1	1:3:5:9	5.130779001	1:7:9:11	4.346773933	1:3:7:17	6.374136829	1:15:27	
1	1:3:5:11	5.171023488	1:5:11:13	4.351437431	1:3:9:23	6.396270913	1:5:7:25	
1	1:3:5:13	5.196152423	1:3:9:27	4.382704973	1:3:11:25	6.433920935	3:5:7	quintal-septimal diminished triad
1	1:3:5:15	5.206811253	1:5:7:21	4.394595457	1:7:21	6.433920935	1:15:21	
1	1:3:5:17	5.217405023	1:3:13:19	4.396829672	1:5:17	6.48261485	1:3:9:23	
1	1:3:5:19	5.229321532	1:11:13	4.429606853	1:5:7:11	6.487154117	1:3:7:13	
1	1:3:5:21	5.248805067	1:3:11:23	4.443096915	1:5:15:27	6.524983829	1:7:15:21	
1	1:3:5:23	5.25914759	1:3:15:17	4.447651627	1:5:7:25	6.533719273	1:7:27	
1	1:3:5:25	5.25914759	1:5:9:17	4.453590356	1:9:17	6.544797457	1:5:11:15	
1	1:3:5:27	5.277632088	1:7:21	4.469338246	1:3:7:19	6.580844365	1:3:5:19	
1	1:3:7:9	5.326586329	1:5:7:23	4.472425879	1:3:11:21	6.605740453	3:5:25	quintal major-minor triad 20:24:25

Commentary on Table 3:

Excerpt from a comparison of the Tenney intersection measure with three variants of harmonic radius, ordering triads and tetrads. As before, the first occurrences of primes are marked in boldface and highlighted yellow. Selected chords from each prime limit up to 19° are highlighted, using distinct colours to distinguish the highest primes (grey = 3°/17°/19°; ivory = 5°; pink = 7°; green = 11°; purple = 13°).

All intersection values for the first 364 chords satisfying the conditions (unique odd partials up to 27°) are equal to 1, thus the values are sorted by increasing numerical values from right to left, first 3-note then 4-note chords. Intersection does not differentiate many of the most commonly occurring chords.

Three radius measures are compared. As in the case of the dyads in Table 2, the inclusion of LCM provides a more balanced otonal/utonal correlation for the simpler and more concordant structures like major and minor triads. Various "common" chords of each prime limit have been given names and voicings in more compact position, to compare the position of these chords in the other lists more easily. Note that the common minor triad 10:12:15 (or 3:5:15 in odd-partial form) occurs early in the LCM list, because the LCM reflects its composition from dyads (intervals), while radius measures its overtonal harmonicity. The first two radius lists have very few chords without 1°.

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"23-LIMIT TUNEABLE INTERVALS below 'A4' – 23-LIMIT TUNEABLE INTERVALS above 'A4'" – tested and notated in three gradations of difficulty (large open notehead = easiest; small black notehead = most difficult) by Marc Sabat (violin/viola) with assistance from Wolfgang von Schweinitz (cello), Beltane Ruiz (bass), Anaïs Chen (violin) – Berlin, 2005 – notated using the Extended Helmholtz-Ellis JI Pitch Notation with cents deviations from 12-tone equal temperament based on A = 0 cents – microtonal accidentals designed by / mikrotonale Vorzeichen konzipiert von Marc Sabat & Wolfgang von Schweinitz, 2004 – PLAINSOUND MUSIC EDITION
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https://en.xen.wiki/w/Chord_complexity

Marc Sabat is a Canadian composer of Ukrainian descent, based in Berlin since 1999. He makes concert and installation pieces, drawing inspiration from investigations of the sounding and perception of just intonation, and relating to various music forms – folk, experimental and classical. He is a frequent collaborator, seeking points of shared exploration and dialogue between various forms of experience and different cultural traditions. http://www.plainsound.org